

## Systolic Computation of Interpolating Polynomials\*

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### Abstract — Zusammenfassung

**Systolic Computation of Interpolating Polynomials.** Several time-optimal and spacetime-optimal systolic arrays are presented for computing a process dependence graph corresponding to the Aitken algorithm. It is shown that these arrays also can be used to compute the generalized divided differences, i.e., the coefficients of the Hermite interpolating polynomial. Multivariate generalized divided differences are shown to be efficiently computed on a 2-dimensional systolic array. The techniques also are applied to the Neville algorithm, producing similar results.

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*Key words:* Newton interpolation, Hermite interpolation, Aitken's algorithm, Neville's algorithm, systolic array

**Systolische Berechnung von interpolierenden Polynomen.** Zur Berechnung eines zum Aitken-Algorithmus gehörigen Prozeßabhängigkeitsgraphen werden einige Zeit-optimale und Raum-Zeit-optimale systolische Felder vorgestellt. Es wird gezeigt, daß man diese Felder auch zur Berechnung verallgemeinerter dividierter Differenzen verwenden kann, wie sie als Koeffizienten des Hermiteschen Interpolationspolynoms auftreten. Die effiziente Berechnung multivariater verallgemeinerter dividierter Differenzen auf einem zweidimensionalen systolischen Feld wird gezeigt. Für den Neville-Algorithmus ergeben die Techniken ähnliche Ergebnisse.

### 1. Introduction

McKeown [12] establishes a rationale for performing iterated interpolation using a systolic array:

“Another consequence of developments in VLSI technology is that interpolation in a table as a means of function approximation could well rehabilitate itself, at least for certain ‘difficult’ functions. . . . the continually falling cost and increasing capacity of semiconductor memory chips could soon make it feasible to consider storing on

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such a chip a table of values for a function requiring complex direct computation. The set of 'chip tables' together with a systolic array for iterated linear interpolation could thus have attractive possibilities."

To obtain a cost-effective design, we must consider the trade-off between the cost of chip tables plus the cost of the systolic array versus the cost of the evaluation of this 'difficult' function at many points using a fast arithmetic processor. However, when we have access to the values of the function at certain points but not to the function itself, systolic arrays will unquestionably provide parallelism. Polynomial interpolation is widely used for curve and surface fitting or, in general, the fitting of multidimensional curves to scattered data. In many applications of the surface-fitting techniques, the aim is to use the data to construct a *contour map* of the unknown function [20]. Since the function is known only at the data points, one must construct a contour map for one of the fitted surfaces. Contour map construction has applications in the oil industry (petroleum explorations), geological maps, and cardiology (heart potentials).

McKeown presents a systolic implementation of Aitken's method of iterated interpolation. In this paper, we consider a systolic version of Newton and Hermite polynomial interpolation using the algorithms of Aitken and Neville. This work builds on that of McKeown by:

1. presenting several alternative systolic implementations of Aitken's algorithm, some of which are optimal;
2. applying these implementations to Neville's algorithm;
3. presenting a 2-level systolic array for computing generalized divided differences;
4. presenting a systolic scheme for the multivariate case;
5. showing that the interpolating polynomials can be implicitly evaluated by only reprogramming the processors.

## 2. Newton and Hermite Polynomial Interpolation

The polynomial interpolation problem is defined as follows:

**Input:**  $n + 1$  pairs of points  $(x_i, f_i) \in F \times F$  for  $0 \leq i \leq n$ , where  $F$  is a field.

**Problem:** Find a polynomial  $p_n(x) \in F[x]$  of degree  $n$  such that  $p_n(x_i) = f_i$  for  $0 \leq i \leq n$ .

The interpolating polynomial of degree  $n$  exists and is unique, provided that  $x_i \neq x_j$  for  $i \neq j$ . Polynomial interpolation is applied to the problem of function approximation. If the points  $f_i$  are the values of a function  $f(x)$  at the points  $x_i$  for  $0 \leq i \leq n$ , then the value  $p_n(\bar{x})$  is used to approximate the value  $f(\bar{x})$  for some  $\bar{x} \neq x_i$  for  $0 \leq i \leq n$ , but which usually is in the same interval as the  $x_i$ s.

The polynomial of degree  $n$  passing through the points  $(x_i, f_i)$  for  $0 \leq i \leq n$  is given in the Newton form as

$$p_n(x) = f_0 + f_{01}(x - x_0) + f_{012}(x - x_0)(x - x_1) + \cdots + f_{012\cdots n}(x - x_0)(x - x_1)\cdots(x - x_{n-1}), \quad (1)$$

where the coefficients  $f_{012\cdots i}$  are called the *divided differences*. The coefficients of the Newton polynomial interpolating the function  $f(x)$  at the node points  $x_i$  for  $0 \leq i \leq n$  can be computed using the well-known algorithms of Neville and Aitken [9, 11]. These algorithms use recursions to compute what is known as the *divided difference table* whose diagonal entries are the desired coefficients of the Newton interpolating polynomial. The Aitken algorithm uses the recursion:

$$f_{012\cdots i,k} = \frac{f_{012\cdots i} - f_{012\cdots i-1,k}}{x_i - x_k} \quad (2)$$

for  $0 \leq i \leq n-1$  and  $i+1 \leq k \leq n$ . The Neville algorithm uses the recursion:

$$f_{i,i+1,\dots,k} = \frac{f_{i,i+1,\dots,k-1} - f_{i+1,i+2,\dots,k}}{x_i - x_k} \quad (3)$$

for  $0 \leq i \leq n-1$  and  $i+1 \leq k \leq n$ .

For Hermite interpolation (the confluent case of Newton interpolation), we construct a polynomial that passes through the values of  $f(x)$ , as well as its derivatives, at the node points  $x_i$ . Let  $\mathbf{m}$  be an ordered collection of  $n$  natural numbers:  $\mathbf{m} = (m_0, m_1, \dots, m_n)$ . We denote the  $k$ th derivative of  $f(x)$  evaluated at  $x_i$  by  $f^{(k)}(x_i)$ , with  $f^{(0)}(x_i) = f(x_i) = f_i$ . If we are given

$$x_i, f_i, f^{(1)}(x_i), f^{(2)}(x_i), \dots, f^{(m_i)}(x_i)$$

for all  $0 \leq i \leq n$ , then we can construct the Hermite polynomial to interpolate this data set. That is, we can construct

$$p_N^{(k,i)}(x_i) = f^{(k,i)}(x_i) \quad \text{for } 0 \leq i \leq n \quad \text{and} \quad 1 \leq k_i \leq m_i.$$

The Hermite interpolating polynomial also is unique, provided  $x_i \neq x_j$  for  $i \neq j$  as is the case for Newton interpolation [2]. The degree of the Hermite interpolating polynomial is

$$N = m_0 + m_1 + \cdots + m_{n-1} + m_n - 1. \quad (4)$$

The Hermite interpolating polynomial can be given as

$$\begin{aligned} p_N(x) = & f_0 + f_{02}(x - x_0) + f_{03}(x - x_0)^2 + \cdots + f_{0m_0}(x - x_0)^{m_0-1} \\ & + f_{0m_01}(x - x_0)^{m_0} + f_{0m_012}(x - x_0)^{m_0}(x - x_1) + \cdots \\ & + f_{0m_01m_1}(x - x_0)^{m_0}(x - x_1)^{m_1-1} + f_{0m_01m_12}(x - x_0)^{m_0}(x - x_1)^{m_1} \\ & + f_{0m_01m_122}(x - x_0)^{m_0}(x - x_1)^{m_1}(x - x_2) + \cdots + \cdots \\ & + f_{0m_01m_12m_2\cdots n m_n}(x - x_0)^{m_0}(x - x_1)^{m_1}(x - x_2)^{m_2}\cdots(x - x_n)^{m_n-1}. \end{aligned} \quad (5)$$

The simplest instance of the Hermite interpolating polynomial is when  $\mathbf{m} = (1, 1, \dots, 1)$ , which corresponds to the Newton interpolating polynomial. The coefficients of (5) are referred to as the *generalized divided differences* [9, 22, 11]. The

Aitken and Neville recursions can be used to compute the generalized divided differences, provided that we avoid division by zero in the recursion formulae (2) and (3) by defining

$$f_{iii\dots i} = \frac{1}{(k-1)!} f^{(k-1)}(x_i), \quad (6)$$

where the subscript  $i$  is repeated  $k$  times [2]. We also denote the generalized divided difference (6) by  $f_{i^k}$ .

The Aitken algorithm for generalized divided differences uses the recursion:

$$f_{0^{a_0} \dots i^{a_i} k^{a_k}} = \frac{f_{0^{a_0} \dots i^{a_i} k^{a_k-1}} - f_{0^{a_0} \dots i^{a_i-1} k^{a_k}}}{x_i - x_k} \quad (7)$$

for  $0 \leq i \leq n-1$  and  $i+1 \leq k \leq n$ ; the corresponding Neville algorithm uses the recursion:

$$f_{i^{a_i} \dots k^{a_k}} = \frac{f_{i^{a_i} \dots k^{a_k-1}} - f_{i^{a_i-1} \dots k^{a_k}}}{x_i - x_k} \quad (8)$$

for  $0 \leq i \leq n-1$  and  $i+1 \leq k \leq n$  where  $a_i \leq m_i$  for  $0 \leq i \leq n$ . The recursion formulae (7) and (8) specialize to (2) and (3) respectively when  $\mathbf{m} = (1, 1, \dots, 1)$ .

### 3. Aitken Algorithm

In this section we give the Aitken algorithm in a Pascal-like notation which allows us to construct its process dependence graph. To illustrate, let  $n = 3$  and  $\mathbf{m} = (2, 2, 2, 2)$ . The input data, then, consists of the function and derivative values  $x_i, f_i, f_{ii}$  for  $0 \leq i \leq 3$ . The following matrices,  $A_{ij} \in F^{m_i \times m_j}$ , contain the generalized divided differences for  $0 \leq i \leq 2$  and  $i < j \leq 3$  as follows:

$$\begin{aligned} A_{03} &= \begin{bmatrix} f_{033} & f_{0033} \\ f_{03} & f_{003} \end{bmatrix}, A_{13} = \begin{bmatrix} f_{00133} & f_{001133} \\ f_{0013} & f_{00113} \end{bmatrix}, A_{23} = \begin{bmatrix} f_{0011233} & f_{00112233} \\ f_{001123} & f_{0011223} \end{bmatrix}, \\ A_{02} &= \begin{bmatrix} f_{022} & f_{0022} \\ f_{02} & f_{002} \end{bmatrix}, A_{12} = \begin{bmatrix} f_{00122} & f_{001122} \\ f_{0012} & f_{00112} \end{bmatrix}, \\ A_{01} &= \begin{bmatrix} f_{011} & f_{0011} \\ f_{01} & f_{001} \end{bmatrix}. \end{aligned}$$

We denote an entry of the above matrices with  $A_{ij}(p, q)$  where  $1 \leq p \leq m_i$  and  $1 \leq q \leq m_j$ . For notational convenience we assume that the index  $p$  increases from left to right, and the index  $q$  increases from bottom to top.

In the general case, the input data is given as  $(x_i, f_{i^k})$  for  $0 \leq i \leq n$  and  $1 \leq k \leq m_i$ . The following procedure given in a Pascal-like notation computes  $A_{ij}(p, q)$  for all  $0 \leq i \leq n-1$ ,  $1 \leq j \leq n$ , and  $1 \leq p \leq m_i$ ,  $1 \leq q \leq m_j$  using recursion (7).

**Procedure Aitken**

*Input:*  $(x_i, f_{ik})$  for  $0 \leq i \leq n$  and  $1 \leq k \leq m_i$ .

*Output:*  $A_{ij}(p, q)$  for  $0 \leq i \leq n-1$ ,  $1 \leq j \leq n$ ,  $1 \leq p \leq m_i$ , and  $1 \leq q \leq m_j$ .

BEGIN

FOR  $i = 0$  TO  $n - 1$  DO

    FOR  $j = i + 1$  TO  $n$  DO

        FOR  $p = 1$  TO  $m_i$  DO

            FOR  $q = 1$  TO  $m_j$  DO

$$A_{ij}(p, q) = \frac{A_{ij}(p, q-1) - A_{ij}(p-1, q)}{x_i - x_j}.$$

    END PROCEDURE

In the above procedure,  $p$  and  $q$  may take the value 0, corresponding to boundary and interface values of the Aitken table. When this happens, that value of the matrix entry is replaced with the appropriate one of the following:

$$A_{0j}(p, 0) = f_{0p} \quad \text{for } 1 \leq p \leq m_0;$$

$$A_{ij}(p, 0) = A_{i-1, i}(m_{i-1}, p) \quad \text{for } 1 \leq i \leq n-1, 2 \leq j \leq n, \text{ and } 1 \leq p \leq m_i;$$

$$A_{0j}(0, q) = f_{jq} \quad \text{for } 1 \leq j \leq n \text{ and } 1 \leq q \leq m_j;$$

$$A_{ij}(0, q) = A_{i-1, j}(m_{i-1}, q), 1 \leq i \leq n-1, 2 \leq j \leq n, \text{ and } 1 \leq q \leq m_j.$$

The outputs of this procedure are the generalized divided differences:

$$A_{ij}(p, q) = f_{0^{m_0}1^{m_1}2^{m_2}\dots i^p j^q}$$

for  $0 \leq i \leq n-1$ ,  $1 \leq j \leq n$ , and  $1 \leq p \leq m_i$ ,  $1 \leq q \leq m_j$ . The coefficients of the Hermite interpolating polynomial are the entries of the matrices  $A_{i-1, i}$  for  $1 \leq i \leq n$ , and expressed in this notation as

$$f_{0^{m_0}1^q} = A_{01}(m_0, q) \quad \text{for } 1 \leq q \leq m_1;$$

$$f_{0^{m_0}1^{m_1}2^q} = A_{12}(m_1, q) \quad \text{for } 1 \leq q \leq m_2;$$

⋮

$$f_{0^{m_0}1^{m_1}2^{m_2}\dots n^q} = A_{n-1, n}(m_{n-1}, q) \quad \text{for } 1 \leq q \leq m_n.$$

#### 4. Process Dependence Graph of the Aitken Algorithm

The data dependences among the entries in the divided difference tables computed by the Aitken and Neville algorithms lend themselves to systolic implementation. In order to exploit the properties of these tables we look at the algorithms to compute the entries. This allows us to form the process dependence graph of each algorithm to compute the divided differences.

We start with  $n = 4$  and  $\mathbf{m} = (1, 1, 1, 1)$ . In this case, the matrices  $A_{ij}$  are simply scalars. In order to see the functional dependence of each entry on the previously computed entries and on the node points  $x_i$  in the table, we use the recursion (2) and fill the Aitken table as follows:

$$\begin{aligned}
 A_{04} &= \frac{f_0 - f_4}{x_0 - x_4} & A_{14} &= \frac{A_{01} - A_{04}}{x_1 - x_4} & A_{24} &= \frac{A_{12} - A_{14}}{x_2 - x_4} & A_{34} &= \frac{A_{23} - A_{24}}{x_3 - x_4} \\
 A_{03} &= \frac{f_0 - f_3}{x_0 - x_3} & A_{13} &= \frac{A_{01} - A_{03}}{x_1 - x_3} & A_{23} &= \frac{A_{12} - A_{13}}{x_2 - x_3} \\
 A_{02} &= \frac{f_0 - f_2}{x_0 - x_2} & A_{12} &= \frac{A_{01} - A_{02}}{x_1 - x_2} \\
 A_{01} &= \frac{f_0 - f_1}{x_0 - x_1}
 \end{aligned}$$

In the denominator terms,  $x_i - x_j$ , the node point  $x_i$  is repeated along a column whereas the node point  $x_j$  is repeated along a row. The positions of the numerator terms, (e.g. the divided difference terms) are similar. First, a divided difference term of the form  $A_{i-1,i}$  is computed on the diagonal, then this term is used in every operation along the  $i$ th column. Based on these observations, we illustrate the process dependence graph of the Aitken algorithm for  $n = 4$  and  $\mathbf{m} = (1, 1, 1, 1)$  in Figure 1. The graph is drawn on the  $(i, j)$  coordinate system. The nodes of this acyclic directed graph represent the operations, and the oriented edges correspond to dependences between the variables used in the operations. The node at point  $(i, j)$  computes  $A_{ij}$  by performing the operation

$$A_{ij} = \frac{A_{i-1,i} - A_{i-1,j}}{x_i - x_j}. \quad (9)$$

A comparison of Figure 1 with the above table reveals the fact that the necessary operands are present for the operations performed in the nodes.

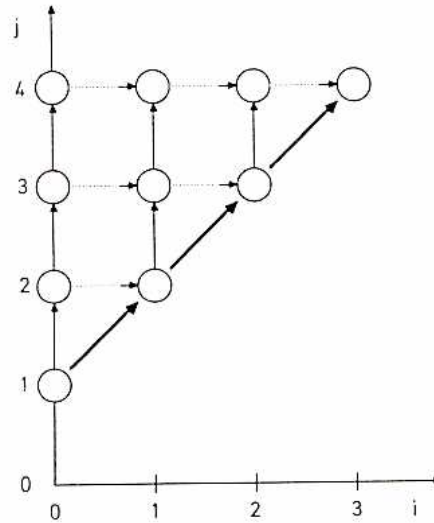


Figure 1. Process dependence graph of the Aitken algorithm,  $G_A$ , for  $n = 4$

These observations can be used to compute the generalized divided differences. In this case we have  $\mathbf{m} = (m_0, m_1, \dots, m_n)$  with  $m_i > 1$  and  $A_{ij}$  are matrices of dimension  $m_i \times m_j$ . If the nodes represent the operations to compute the entries of the matrix  $A_{ij}$ , then the process dependence graph for the generalized divided difference tables is the same as the Newton case, illustrated in Figure 1. Let  $m_i = 3$  and  $m_j = 4$ . By using Procedure Aitken, these entries are computed as follows:

$$\begin{aligned}
 A_{ij}(1, 4) &= \frac{A_{ij}(1, 3) - A_{i-1, j}(m_{i-1}, 4)}{x_i - x_j} & A_{ij}(2, 4) &= \frac{A_{ij}(2, 3) - A_{i, j}(1, 4)}{x_i - x_j} \\
 A_{ij}(1, 3) &= \frac{A_{ij}(1, 2) - A_{i-1, j}(m_{i-1}, 3)}{x_i - x_j} & A_{ij}(2, 3) &= \frac{A_{ij}(2, 2) - A_{i, j}(1, 3)}{x_i - x_j} \\
 A_{ij}(1, 2) &= \frac{A_{ij}(1, 1) - A_{i-1, j}(m_{i-1}, 2)}{x_i - x_j} & A_{ij}(2, 2) &= \frac{A_{ij}(2, 1) - A_{i, j}(1, 2)}{x_i - x_j} \\
 A_{ij}(1, 1) &= \frac{A_{i-1, i}(m_{i-1}, 1) - A_{i-1, j}(m_{i-1}, 1)}{x_i - x_j} & A_{ij}(2, 1) &= \frac{A_{i-1, i}(m_{i-1}, 2) - A_{i, j}(1, 1)}{x_i - x_j} \\
 A_{ij}(3, 4) &= \frac{A_{ij}(3, 3) - A_{i, j}(2, 4)}{x_i - x_j} \\
 A_{ij}(3, 3) &= \frac{A_{ij}(3, 2) - A_{i, j}(2, 3)}{x_i - x_j} \\
 A_{ij}(3, 2) &= \frac{A_{ij}(3, 1) - A_{i, j}(2, 2)}{x_i - x_j} \\
 A_{ij}(3, 1) &= \frac{A_{i-1, i}(m_{i-1}, 3) - A_{i, j}(2, 1)}{x_i - x_j}
 \end{aligned}$$

The process dependence graph of the above operations is illustrated in Figure 2. The node at the point  $(i, j)$  here represents  $m_i \times m_j$  smaller nodes connected in a rectangular mesh where the entries of the matrix  $A_{ij}$  are computed. If  $m_i = 1$  for all  $0 \leq i \leq n$ , then this graph consists of only one node and the process dependence graph of the generalized Aitken table reduces to the simple case depicted in Figure 1. We thus can view Figure 1 as the process dependence graph of the general case, where the nodes represent rectangular meshes of simpler nodes (operations). This hierarchical view simplifies the treatment of each case, and allows us to embed the process dependence graph in spacetime, producing various systolic arrays. In the following sections, the process dependence graph,  $G_A$ , for Aitken's algorithm is embedded in spacetime, obtaining several different systolic arrays (for spacetime embedding techniques, see [14, 15, 3, 13, 16, 4, 17]).

## 5. The McKeown Array

In this section, we present McKeown's array [12]. We embed the process dependence graph for Aitken's algorithm in spacetime. The abscissa is interpreted as time

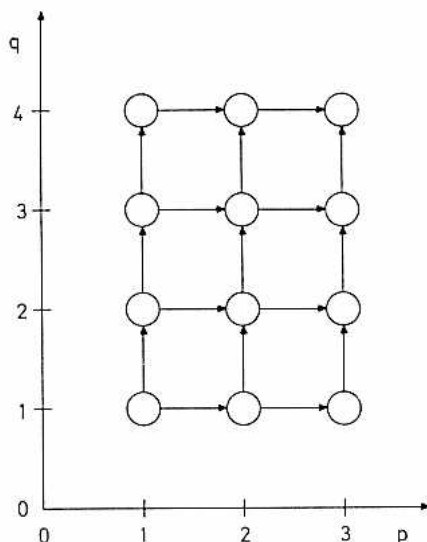


Figure 2. Process dependence graph of the Aitken algorithm to compute the generalized divided differences for  $m_i = 3$  and  $m_j = 4$

( $t$ ); the ordinate as space ( $s$ ). The linear embedding,  $E_1$ , is as follows:

$$\begin{bmatrix} t \\ s \end{bmatrix} = T_1 \begin{bmatrix} i \\ j \end{bmatrix} \quad \text{where} \quad T_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The result, depicted in Figure 3 for  $n = 6$ , is McKeown's array. Data that flows south  $\rightarrow$  north in Figure 1, flows in the direction of time (perpendicular to space) in the McKeown design: It is in the processors' memory. Data that flows west  $\rightarrow$  east in Figure 1, flows up through the McKeown array. Data that flows south  $\rightarrow$  east in

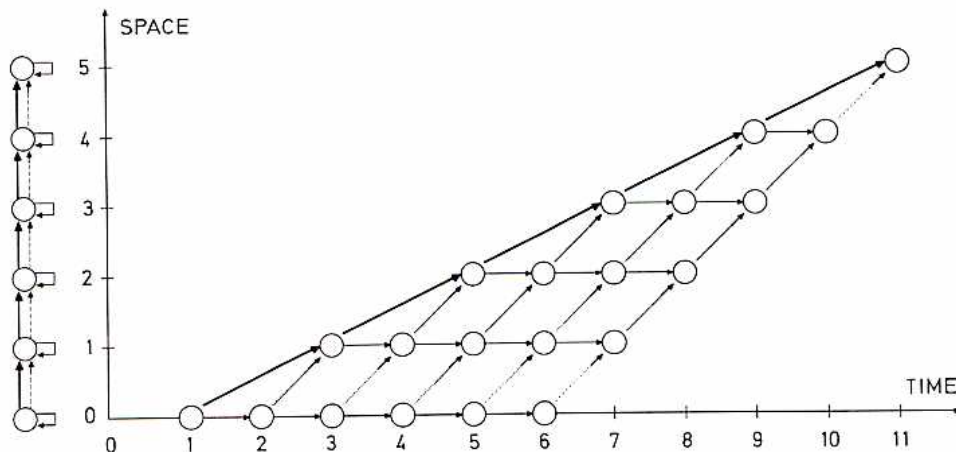


Figure 3. A spacetime representation of McKeown's array, for  $n = 6$ . Its spatial projection is a linear array of  $n$  processors



Figure 1, also flows up through the McKeown array, but at half the speed of the west  $\rightarrow$  east data.

Process  $(i, j)$  is executed at time  $i + j$  in processor  $j$ . By inspection, we see that the array uses  $n$  processors, finishing the computation in  $2n - 1$  steps. The number of vertices (processes) in a longest directed path in any process dependence graph is a lower bound on the number of steps of any schedule for computing the processes. In our graph, the number of vertices in a longest path is  $2n - 1$ . McKeown's array thus uses a spacetime embedding that is optimal with respect to the number of steps used. Such an embedding is referred to as *time-optimal*.

### 6. A Spacetime-Optimal Version of the McKeown Array

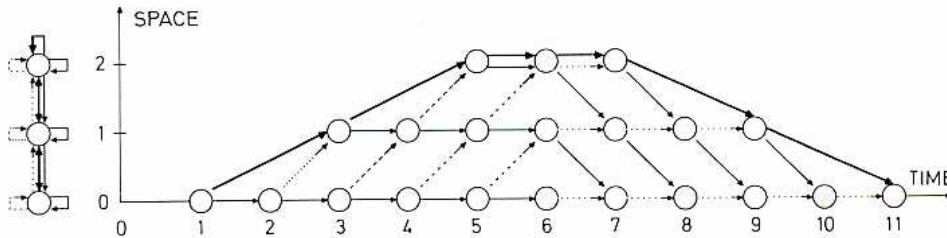
**Definition 1** A graph's embedding is spacetime-optimal when it is space-minimal among those embeddings that are time-optimal.

We now make a slight modification to the McKeown array, producing a *spacetime-optimal* array. There is unused time on the lower numbered processors. We reschedule the computation done on the upper processors onto these lower processors. More formally, we embed the process dependence graph as follows:

$$\begin{aligned} \begin{bmatrix} t \\ s \end{bmatrix} &:= T_1 \begin{bmatrix} i \\ j \end{bmatrix} & \text{for } i + j \leq n; \\ \begin{bmatrix} t \\ s \end{bmatrix} &:= \begin{bmatrix} i + j \\ n - j \end{bmatrix} & \text{for } i + j > n. \end{aligned}$$

This embedding,  $E_2$ , is illustrated, for  $n = 6$ , in Figure 4. This design has 2 phases of data movement. In the 1st phase, data moves as in the McKeown design. As the 1st phase ends, and the 2nd begins, there is a transition: When  $n$  is even (as depicted in Figure 4), there are 2 time steps in which the south  $\rightarrow$  east data flows in the direction of time: It is in the uppermost processor's memory for these 2 steps. When  $n$  is odd, the south  $\rightarrow$  east data always moves through the array.

In the 2nd phase, data moves as follows. Data that flows south  $\rightarrow$  north in Figure 1, flows down through the array. Data that flows west  $\rightarrow$  east in Figure 1, flows in the direction of time: It is remembered in this phase. Data that flows south  $\rightarrow$  east



**Figure 4.** Spacetime-optimal embedding  $E_2$  of  $G_A$ , a process dependence graph for Aitken's algorithm. Its spatial projection is a linear array of  $\lfloor n/2 \rfloor$  processors

in Figure 1, also flows down through this array, but at half the speed of the south  $\rightarrow$  north data.

Of those spacetime embeddings that are time-optimal, this embedding also is space-minimal:

**Theorem 1** *Embedding  $E_2$  of  $G_A$  is spacetime-optimal.*

*Proof:* The embedding  $E_2$  is identical to  $E_1$  with respect to time: it too is time-optimal. We argue space minimality as follows. To reduce space, we must reschedule a processor's processes to other processors. Let us focus on the points in time where all processors are used. In Figure 4, they occur for  $t = 5, 6, 7$ . During these points all 3 processors are used. We refer to such points in time as the *space-maximal* points of the embedding. In order to reduce the spatial extent of this embedding, it is necessary that the processes from some processor's space-maximal points be rescheduled onto other processors. We can reduce the spatial extent of the embedding depicted in Figure 4, for example, if processes embedded at coordinates  $(5, 3)$ ,  $(6, 3)$ , and  $(7, 3)$  can be rescheduled onto other processors. Notice that these processes are on a longest directed path in the process dependence graph. This means that none can be rescheduled for earlier completion without violating an order constraint. Neither can they be scheduled for later completion without either violating an order constraint or extending the overall completion time, violating the time-optimality property. In fact, in this embedding every process is on some longest path, and hence can be rescheduled onto neither an earlier nor a later cycle. For this process dependence graph, processes occurring during the space-maximal points, in particular, cannot be rescheduled. Therefore the number of processors used cannot be reduced without violating time-optimality. We conclude that the embedding is spacetime-optimal: Any spacetime embedding of this process dependence graph that completes in  $2n - 1$  cycles, must use at least  $\lceil \frac{n}{2} \rceil$  processors.  $\square$

Moreover, the nonlinearity of our spacetime transformation is necessary: There does not exist a linear embedding of the initial indices that is spacetime-optimal.

## 7. Some Other Spacetime-Optimal Arrays

We now present 2 other spacetime-optimal embeddings of the process dependence graph of Figure 1. The first is another variation of McKeown's array. We again reschedule the computation done on the upper processors onto the lower processors. To do this, we connect the endpoints of the linear array, making a *ring* of processors. More formally, we nonlinearly embed the process dependence graph as follows:

$$t := i + j;$$

$$s := i \bmod \left\lfloor \frac{n}{2} \right\rfloor.$$

This embedding,  $E_3$ , is illustrated, for  $n = 6$ , in Figure 5. This design has data flow characteristics that are identical to the McKeown array, except that the upper processor is attached to the lower processor, and data movement wraps around.

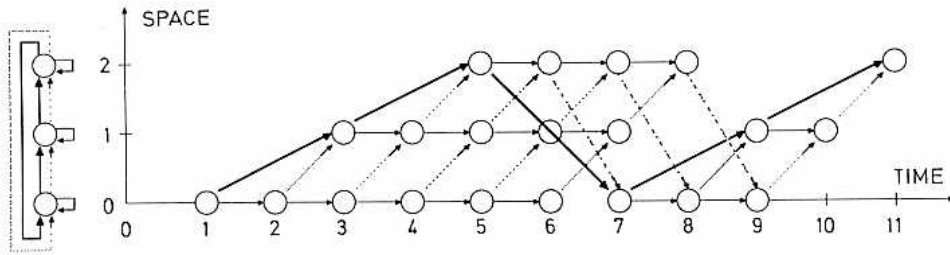


Figure 5. Spacetime-optimal embedding  $E_3$  of  $G_4$ , a process dependence graph for Aitken's algorithm. Its spatial projection is a ring of  $\lceil n/2 \rceil$  processors

Since this embedding results in a computation of the process dependence graph that uses  $2n - 1$  steps and  $\lceil \frac{n}{2} \rceil$  processors, it too is spacetime-optimal.

Finally, we present a bilateral array in which the south  $\rightarrow$  north data of Figure 1 moves up through the array, while the west  $\rightarrow$  east data moves down through the array. Such a data movement scheme may be useful, depending on the larger context of which this computation is a part. The spacetime embedding,  $E_4$ , is presented in 2 steps:

1. First, we embed the process dependence graph, illustrated in Figure 6, as follows:

$$\begin{bmatrix} t \\ s \end{bmatrix} := T_2 \begin{bmatrix} i \\ j - 1 \end{bmatrix} \quad \text{where} \quad T_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

In this spacetime embedding, when processors are used, they are used every other time step.

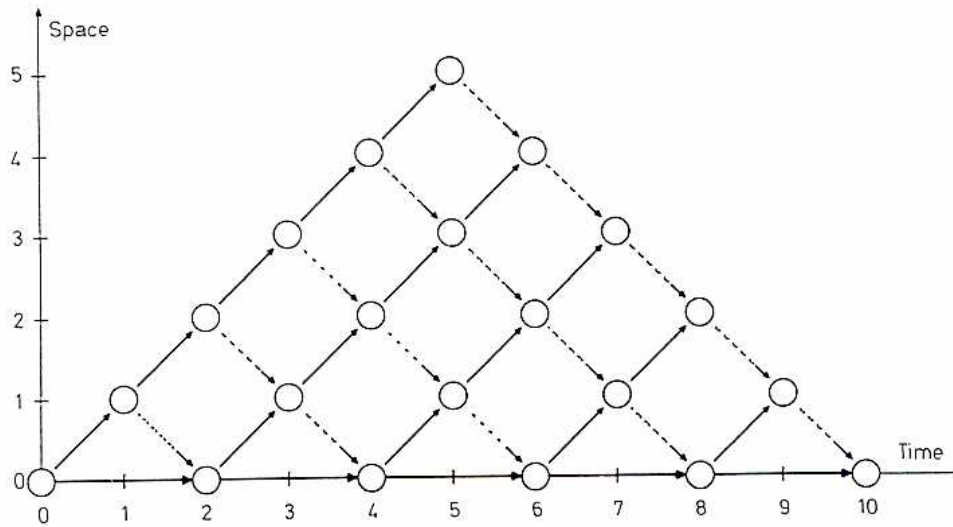
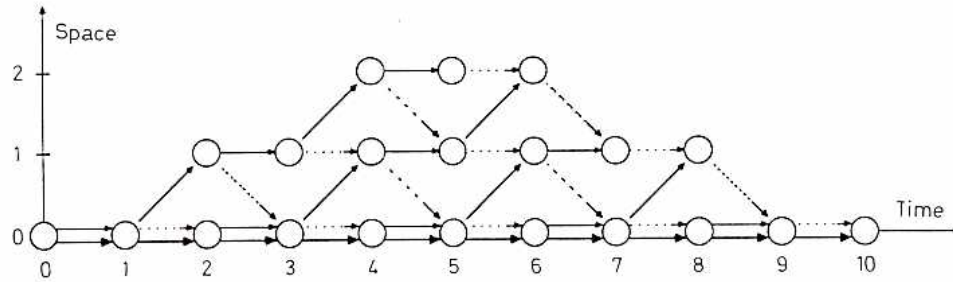


Figure 6. An embedding of a process dependence graph for Aitken's algorithm. Its spatial projection is a linear array of  $n$  processors



**Figure 7.** A spacetime-optimal embedding of  $G_A$ , a process dependence graph for Aitken's algorithm. Its spatial projection is a linear array of  $\lceil n/2 \rceil$  processors

2. We now compress the spatial extent of this embedding with the following nonlinear transformation:

$$T_2 = \lfloor C \rfloor,$$

where

$$C = \begin{bmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{bmatrix}, \quad \text{and} \quad y = \lfloor C \rfloor x = \lfloor Cx \rfloor, \quad \text{where} \quad \left\lfloor \begin{bmatrix} i \\ j \end{bmatrix} \right\rfloor = \begin{bmatrix} \lfloor i \rfloor \\ \lfloor j \rfloor \end{bmatrix}.$$

Processor efficiency (i.e., the percentage of time that a processor is used) is doubled asymptotically by this nonlinear transformation. Figure 7 illustrates the result.

This design has 2 phases of data movement which alternate with each time step. Regardless of the phase, data that flows south  $\rightarrow$  east in Figure 1, flows in the direction of time: It is remembered.

In phase *A*, data that flows south  $\rightarrow$  north in Figure 1, flows up through the array; data that flows west  $\rightarrow$  east in Figure 1, flows in the direction of time.

In phase *B*, data that flows south  $\rightarrow$  north in Figure 1, flows in the direction of time; data that flows west  $\rightarrow$  east in Figure 1, flows up through the array.

Since this second transformation results in an embedding that uses  $2n - 1$  steps and  $\lceil \frac{n}{2} \rceil$  processors, it too is spacetime-optimal.

## 8. Two-Dimensional Array

We now present a  $2 - d$  array for computing the process dependence graph of Figure 1. This is done by embedding the process dependence graph into a  $3 - d$  space. One way to do this is with a linear embedding,  $E_5$ :

$$\begin{bmatrix} t \\ s_1 \\ s_2 \end{bmatrix} := T_3 \begin{bmatrix} i \\ j \end{bmatrix} \quad \text{where} \quad T_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

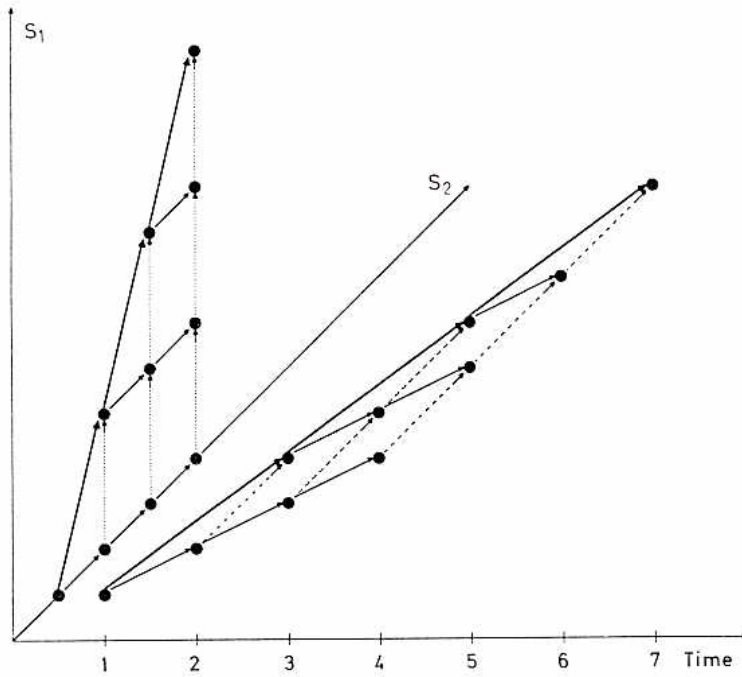


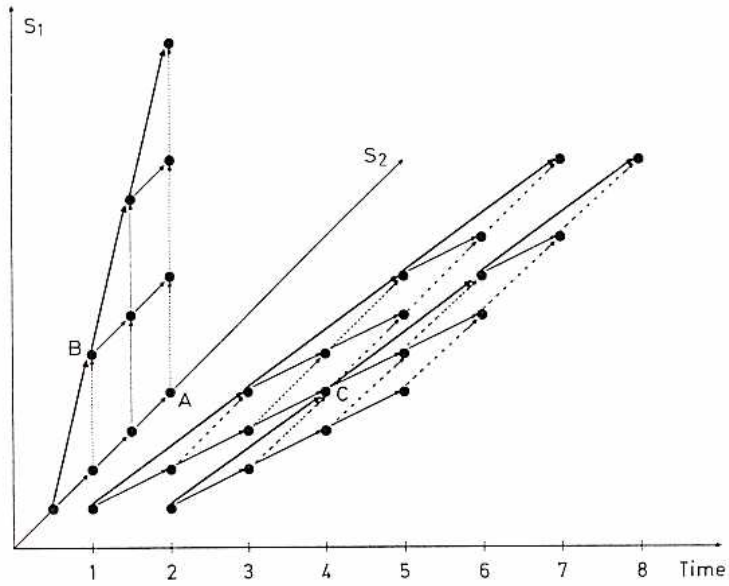
Figure 8. An embedding of  $G_A$  whose spatial projection is a triangular array of processors

Figure 8 illustrates the result. In this array, there is a processor for every process (in the process dependence graph). The flow of data between processors corresponds to the arcs in the process dependence graph. Every processor whose corresponding vertex in Figure 1 has indices whose sum is  $k$  executes its process at step  $k$ . Execution completes after  $2n - 1$  steps. This embedding has the property that each processor is used exactly once per execution of the process dependence graph. The array can start executing a new process dependence graph every step. Figure 9 is intended to illustrate the pipeline quality of this array; it shows 2 process dependence graphs embedded in spacetime such that execution of the 2nd starts 1 step after the 1st. Consequently, executing  $k$  such process dependence graphs uses  $2n + k - 2$  steps.

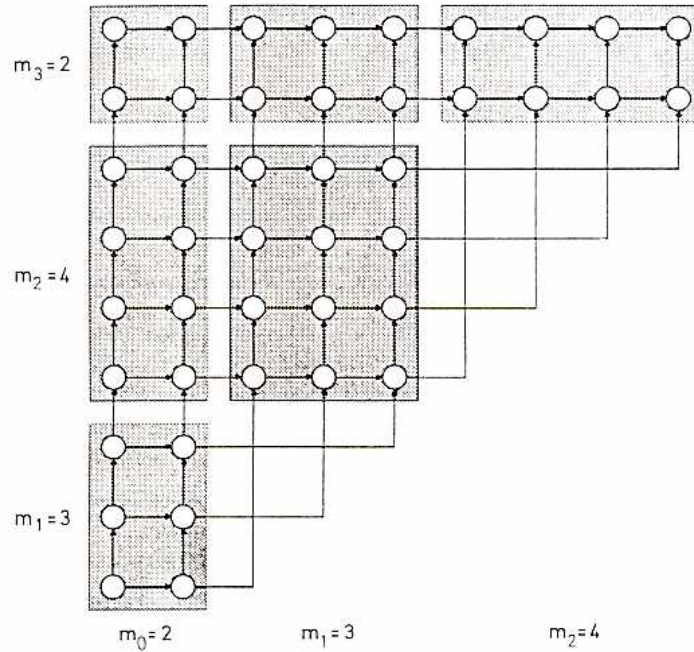
The  $2 - d$  systolic array is useful when one is computing many interpolation polynomials with possibly different set of points. Such is the case for multivariate interpolation, which we consider in §11.

### 9. Computing Generalized Divided Differences

In this section, we incorporate extensions to Aitken's algorithm, enabling it to be used to compute generalized divided differences. As mentioned in §4, the process dependence graph for the generalized divided differences has a two-level structure. At the top level, its structure is that of the divided differences process dependence graph, illustrated in Figure 1. Each node in this top level structure comprises a



**Figure 9.** An embedding of 2 copies of  $G_A$ . There are 2 distinct processes that appear as though they are embedded in the same point (C) of spacetime. They, in fact, are embedded in distinct spatial coordinates. The process from the 1st copy of  $G_A$  executes on processor A while the process from the 2nd copy of  $G_A$  executes on processor B



**Figure 10.** A 2-level process dependence graph,  $G_{G_A}$ , for computing generalized divided differences

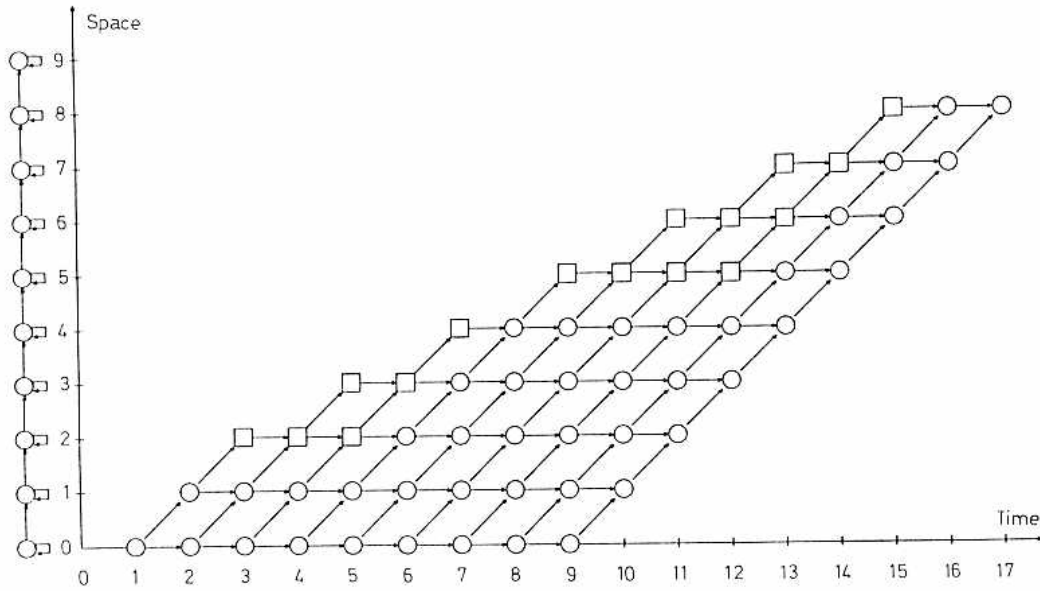


Figure 11. The McKewon embedding,  $E_1$ , applied to  $G_{GA}$ , a 2-level process dependence graph for computing generalized divided differences

Table 1. Complexities of various embeddings of  $G_{GA}$ , a 2-level process dependence graph for computing generalized divided differences

Embedding	Time Complexity	Space Complexity
McKewon ( $E_1$ )	$m_0 + m_n - 1 + 2 \sum_{i=1}^{n-1} m_i$	$\sum_{i=0}^{n-1} m_i$
Optimal McKewon ( $E_2$ )	$m_0 + m_n - 1 + 2 \sum_{i=1}^{n-1} m_i$	$\lceil \frac{1}{2} \sum_{i=0}^{n-1} m_i \rceil$
Optimal Ring ( $E_3$ )	$m_0 + m_n - 1 + 2 \sum_{i=1}^{n-1} m_i$	$\lceil \frac{1}{2} \sum_{i=0}^{n-1} m_i \rceil$
Optimal Bilateral ( $E_4$ )	$m_0 + m_n - 1 + 2 \sum_{i=1}^{n-1} m_i$	$\lceil \frac{1}{2} [\sum_{i=0}^{n-1} m_i - 1] \rceil$
2-d ( $E_5$ )	$m_0 + m_n - 1 + 2 \sum_{i=1}^{n-1} m_i$	$\sum_{i=0}^{n-1} \sum_{j=i+1}^n m_i m_j$

rectangular mesh of nodes that perform scalar operations, illustrated in Figure 2. Figure 10 illustrates how these rectangular meshes are interconnected to form the 2-level process dependence graph  $G_{GA}$ . The spacetime embeddings used for computing the divided differences (see §5–8) can be applied to the process dependence graph for computing generalized divided differences. The McKewon embedding for the generalized case is illustrated in Figure 11. Table 1 contains the time and space complexities of these embeddings. Embeddings  $E_2$ ,  $E_3$ , and  $E_4$  also are spacetime-optimal for the process dependence graph,  $G_{GA}$ , that computes generalized divided differences.

There are several other ways to embed this 2-level process dependence graph. Some of these design alternatives are touched upon in the Conclusion.

### 10. Neville Algorithm and Its Process Dependence Graph

In this section we describe the Neville algorithm and its process dependence graph. For simplicity of illustration, we assume that  $n = 3$  and  $\mathbf{m} = (2, 2, 2, 2)$ . Analogous to the Aitken algorithm, we define the matrices  $N_{ij} \in F^{m_i \times m_j}$  for  $0 \leq i \leq 2$  and  $i < j \leq 3$  as follows:

$$N_{03} = \begin{bmatrix} f_{00112233} & f_{0112233} \\ f_{0011223} & f_{011223} \end{bmatrix}, N_{13} = \begin{bmatrix} f_{112233} & f_{12233} \\ f_{11223} & f_{1223} \end{bmatrix}, N_{23} = \begin{bmatrix} f_{2233} & f_{233} \\ f_{223} & f_{23} \end{bmatrix},$$

$$N_{02} = \begin{bmatrix} f_{001122} & f_{01122} \\ f_{00112} & f_{0112} \end{bmatrix}, N_{12} = \begin{bmatrix} f_{1122} & f_{122} \\ f_{112} & f_{12} \end{bmatrix},$$

$$N_{01} = \begin{bmatrix} f_{0011} & f_{011} \\ f_{001} & f_{01} \end{bmatrix}.$$

In general we have

$$N_{ij}(p, q) = f_{i^p(i+1)^{m_{i+1}} \cdots (j-1)^{m_{j-1}} j^q}$$

where index  $p$  is assumed to increase from right to left and index  $q$  from bottom to top. Here, the diagonal entries,  $N_{01}$ ,  $N_{12}$ , and  $N_{23}$ , are computed first, followed by the entries above the main diagonal,  $N_{02}$ , and  $N_{13}$ , and so on. This process is formalized in the following procedure.

#### Procedure Neville

*Input:*  $(x_i, f_{i^k})$  for  $0 \leq i \leq n$  and  $1 \leq k \leq m_i$ .

*Output:*  $N_{ij}(p, q)$  for  $0 \leq i \leq n-1$ ,  $1 \leq j \leq n$ ,  $1 \leq p \leq m_i$ , and  $1 \leq q \leq m_j$ .

BEGIN

FOR  $j = 0$  TO  $n$  DO

FOR  $i = 0$  TO  $n - j$  DO

FOR  $p = 1$  TO  $m_i$  DO

FOR  $q = 1$  TO  $m_j$  DO

$$N_{i,i+j}(p, q) = \frac{N_{i,i+j}(p, q-1) - N_{i,i+j}(p-1, q)}{x_i - x_{i+j}}.$$

END PROCEDURE

The boundary and interface conditions for the Neville Procedure are found to be

$$N_{i,i+1}(p, 0) = f_{i^p} \quad \text{for } 0 \leq i \leq n-1 \quad \text{and } 1 \leq p \leq m_0;$$

$$N_{i,i+j}(p, 0) = N_{i,i+j-1}(p, m_{i+j-1}) \quad \text{for } 2 \leq j \leq n, 0 \leq i \leq n-j, \text{ and } 1 \leq p \leq m_i;$$

$$N_{i,i+1}(0, q) = f_{(i+1)^q} \quad \text{for } 0 \leq i \leq n-1 \quad \text{and } 1 \leq q \leq m_{i+1};$$

$$N_{i,i+j}(0, q) = N_{i,i+j+1}(m_{i+j+1}, q) \quad \text{for } 2 \leq j \leq n, 0 \leq i \leq n-j,$$

$$\text{and } 1 \leq q \leq m_{i+j+1}.$$

To illustrate, we take  $n = 4$  and  $\mathbf{m} = (1, 1, 1, 1, 1)$ .



$$\begin{aligned}
 N_{04} &= \frac{N_{03} - N_{14}}{x_0 - x_4} & N_{14} &= \frac{N_{13} - N_{24}}{x_1 - x_4} & N_{24} &= \frac{N_{23} - N_{34}}{x_2 - x_4} & N_{34} &= \frac{f_3 - f_4}{x_3 - x_4} \\
 N_{03} &= \frac{N_{02} - N_{13}}{x_0 - x_3} & N_{13} &= \frac{N_{12} - N_{23}}{x_1 - x_3} & N_{23} &= \frac{f_2 - f_3}{x_2 - x_3} \\
 N_{02} &= \frac{N_{01} - N_{12}}{x_0 - x_2} & N_{12} &= \frac{f_1 - f_2}{x_1 - x_2} \\
 N_{01} &= \frac{f_0 - f_1}{x_0 - x_1}
 \end{aligned}$$

For the denominator terms  $x_i - x_j$ , the point  $x_i$  is repeated along a column whereas the point  $x_j$  is repeated along a row. Also when a divided difference term,  $N_{ij}$ , is computed, then this term is used to compute  $N_{i-1,j}$  and  $N_{i,j+1}$ . The process dependence graph of the Neville algorithm,  $G_N$ , is given in Figure 12 drawn on the  $(i, j)$  coordinate system where the node at the point  $(i, j)$  computes  $N_{ij}$ , performing the operation

$$N_{ij} = \frac{N_{i,j-1} - N_{i+1,j}}{x_i - x_j} \tag{10}$$

For the generalized divided differences, similar to the Aitken case, we have  $\mathbf{m} = (m_0, m_1, \dots, m_n)$  with  $m_i > 1$  and  $N_{ij}$  are matrices of dimension  $m_i \times m_j$ . Now we let the node in Figure 12 represent the set operations to compute the entries of the matrix  $N_{ij}$ . Figure 12, then, represents the process dependence graph for the generalized divided difference table as well. If  $m_i = 3$  and  $m_j = 4$ , then by using Procedure Neville we compute the entries of  $N_{ij}$  as

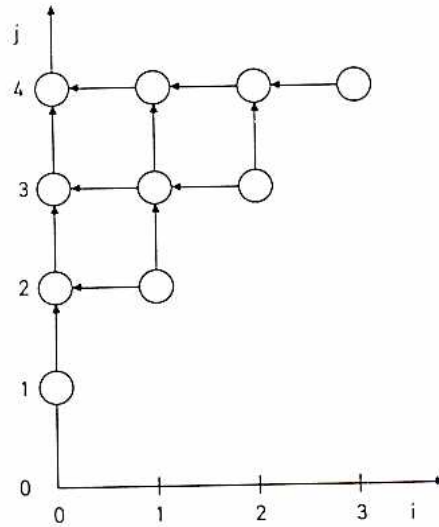


Figure 12. Process dependence graph of the Neville algorithm,  $G_N$ , for  $n = 4$

$$N_{ij}(3, 4) = \frac{N_{ij}(3, 3) - N_{ij}(2, 4)}{x_i - x_j}$$

$$N_{ij}(2, 4) = \frac{N_{ij}(2, 3) - N_{ij}(1, 4)}{x_i - x_j}$$

$$N_{ij}(3, 3) = \frac{N_{ij}(3, 2) - N_{ij}(2, 3)}{x_i - x_j}$$

$$N_{ij}(2, 3) = \frac{N_{ij}(2, 2) - N_{ij}(1, 3)}{x_i - x_j}$$

$$N_{ij}(3, 2) = \frac{N_{ij}(3, 1) - N_{ij}(2, 2)}{x_i - x_j}$$

$$N_{ij}(2, 2) = \frac{N_{ij}(2, 1) - N_{ij}(1, 2)}{x_i - x_j}$$

$$N_{ij}(3, 1) = \frac{N_{i,j-1}(3, m_{j-1}) - N_{ij}(2, 1)}{x_i - x_j}$$

$$N_{ij}(2, 1) = \frac{N_{i,j-1}(2, m_{j-1}) - N_{ij}(1, 1)}{x_i - x_j}$$

$$N_{ij}(1, 4) = \frac{N_{ij}(1, 3) - N_{i+1,j}(m_{i+1}, 4)}{x_i - x_j}$$

$$N_{ij}(1, 3) = \frac{N_{ij}(1, 2) - N_{i+1,j}(m_{i+1}, 3)}{x_i - x_j}$$

$$N_{ij}(1, 2) = \frac{N_{ij}(1, 1) - N_{i+1,j}(m_{i+1}, 2)}{x_i - x_j}$$

$$N_{ij}(1, 1) = \frac{N_{i,j-1}(1, m_{j-1}) - N_{i+1,j}(m_{i+1}, 1)}{x_i - x_j}$$

The process dependence graph of the above operations is illustrated in Figure 13. Similar to the Aitken algorithm, the node at the point  $(i, j)$  represents  $m_i \times m_j$  smaller nodes connected in a rectangular mesh.

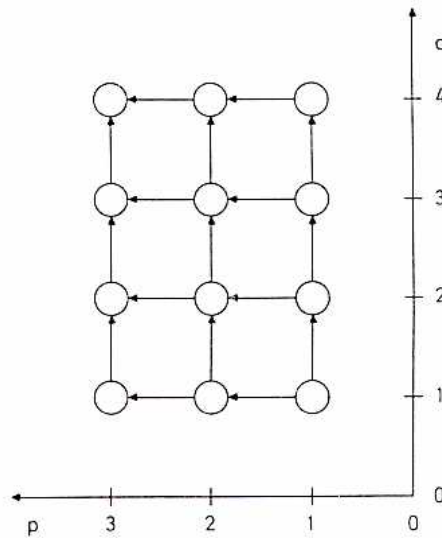


Figure 13. Process dependence graph of the Neville algorithm to compute the generalized divided differences for  $m_i = 3$  and  $m_j = 4$

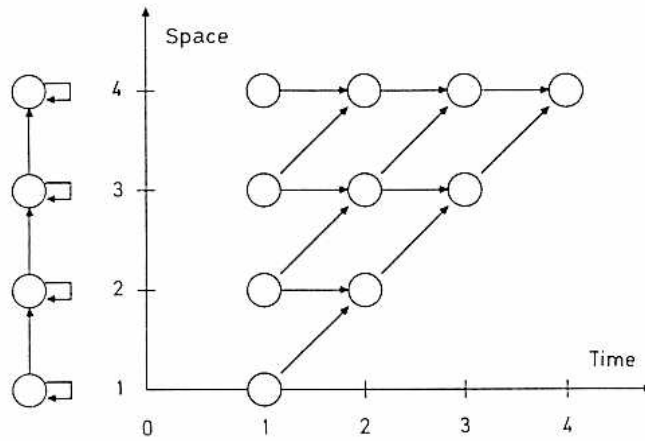


Figure 14. An embedding of a process dependence graph of the Neville algorithm,  $G_N$ , for  $n = 4$ . Its spatial projection is a linear array of  $n$  processors

The Neville algorithm's process dependence graph can be embedded in spacetime in a way that is similar to the embeddings of the Aitken algorithm described in §5–8. Here we describe one such embedding which is a variation of McKeown's array. The spacetime embedding of the process dependence graph given in Figure 12 is as follows:

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}.$$

This embedding is illustrated in Figure 14 for  $n = 4$ . Data that flows south  $\rightarrow$  north in Figure 12, flows in the direction of time and space. It flows up through the array. Data that flows east  $\rightarrow$  west in Figure 12, flows in the direction of time: it stays in the processors' memory. We see that the array uses  $n$  processors, finishing the computation in  $n$  steps.

## 11. Computation of Multivariate Divided Differences

In this section we consider the problem of multivariate interpolation. We illustrate the underlying algorithms and systolic arrays for computation of the bivariate divided differences. Extension to more than two variables is straightforward.

The general problem of multivariate interpolation of an arbitrary set of points is difficult, and the algorithms are complicated. The main reason for this is that *unisolvence* cannot be satisfied for arbitrarily spaced data [2]. In one dimension, the distribution of the points is restricted enough to satisfy unisolvence [21]. In more than one dimension, one usually needs to make assumptions regarding the distribution of the points (see, e.g., [18, 8, 7]). Otherwise it is required that certain determinants which appear in constructing the interpolating polynomials do not vanish

[19]. We consider bivariate interpolation only on a two-dimensional grid. This case is common, and also offers a natural generalization of the divided difference algorithms to higher dimensions. The machinery developed so far for the computation of the univariate divided differences is well suited to this case. We assume that there exists a bivariate function  $f(x, y)$  evaluated at  $(n + 1)(n + 1)$  points on the Euclidean plane  $(x_i, y_j)$  for  $0 \leq i, j \leq n$ . Denote  $f(x_i, x_j)$  by  $f_{i,j}$ ; then the points are arranged as follows:

$$\begin{array}{cccccc} f_{0,n} & f_{1,n} & f_{2,n} & \cdots & f_{n,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{0,2} & f_{1,2} & f_{2,2} & \cdots & f_{n,2} \\ f_{0,1} & f_{1,1} & f_{2,1} & \cdots & f_{n,1} \\ f_{0,0} & f_{1,0} & f_{2,0} & \cdots & f_{n,0} \end{array}$$

The bivariate Newton polynomial interpolating this data set can be given as

$$P(x, y) = \sum_{i=0}^n \sum_{j=0}^n (x - x_0) \cdots (x - x_{i-1})(y - y_0) \cdots (y - y_{j-1}) f_{01 \cdots i, 01 \cdots j}$$

where the coefficients  $f_{01 \cdots i, 01 \cdots j}$  are bivariate divided differences, found by applying the univariate divided difference algorithms repeatedly in the  $x$  and  $y$  directions. We first simultaneously apply the Aitken (or the Neville) Algorithm in the  $x$  direction for all  $0 \leq j \leq n$ , finding

$$\begin{array}{cccccc} f_{0,n} & f_{01,n} & f_{012,n} & \cdots & f_{012 \cdots n,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{0,2} & f_{01,2} & f_{012,2} & \cdots & f_{012 \cdots n,2} \\ f_{0,1} & f_{01,1} & f_{012,1} & \cdots & f_{012 \cdots n,1} \\ f_{0,0} & f_{01,0} & f_{012,0} & \cdots & f_{012 \cdots n,0} \end{array}$$

The application of the Aitken (or the Neville) Algorithms in the  $y$  direction for all  $0 \leq i \leq n$  yields the bivariate divided differences:

$$\begin{array}{cccccc} f_{0,012 \cdots n} & f_{01,012 \cdots n} & f_{012,012 \cdots n} & \cdots & f_{012 \cdots n,012 \cdots n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{0,012} & f_{01,012} & f_{012,012} & \cdots & f_{012 \cdots n,012} \\ f_{0,01} & f_{01,01} & f_{012,01} & \cdots & f_{012 \cdots n,01} \\ f_{0,0} & f_{01,0} & f_{012,0} & \cdots & f_{012 \cdots n,0} \end{array}$$

The 2-d array developed in §8 can be used to perform the above computations efficiently. Assume initially that  $B_{ij} = f_{i,j}$  for  $0 \leq i, j \leq n$ . The following procedure computes the bivariate divided differences using a 2-d Aitken array that is attached to a host computer.

```

Procedure Bivariate ( $x_i, y_j, B_{ij}; 0 \leq i, j \leq n$ )
BEGIN
FOR  $j = 0$  TO  $n$  DO
   $B_{ij} = \text{Univariate}$  ( $x_i, B_{ij}; 0 \leq i \leq n$ ).
FOR  $i = 0$  TO  $n$  DO
   $B_{ij} = \text{Univariate}$  ( $y_i, B_{ij}; 0 \leq j \leq n$ ).
END PROCEDURE

```

At the end of all computations the arrays  $B_{ij}$  hold the bivariate divided differences:  $B_{ij} = f_{01\dots i, 01\dots j}$ , for  $0 \leq i, j \leq n$ .

The execution of  $k$  process dependence graphs on a 2-d Aitken array takes  $2n + k - 2$  time steps where each step is a divided difference operation (9). Thus, the above procedure uses  $3n - 1$  steps to execute the first loop, and the same number of steps for the second loop; which gives the total parallel time as  $T_{par} = 6n - 2$ . Since the sequential computation of the bivariate divided differences takes  $T_{seq} = n(n - 1)(n + 1)$ , and 2-d array contains  $p = \frac{n(n - 1)}{2}$  processing elements, the efficiency of this scheme will be

$$E = \frac{T_{seq}}{pT_{par}} = \frac{n + 1}{3n - 1} \approx 0.33,$$

which is asymptotically optimal. Computation of bivariate (or *multivariate*) generalized divided differences can be done by a straightforward extension of the above algorithm.

## 12. Discussion and Conclusion

The graph used to represent the Aitken algorithm is very similar to that of the Neville algorithm. Since both graphs have a triangular shape, their space-time embeddings are similar.

We have presented several systolic designs for implementing the Aitken and the Neville algorithms. There are, in fact, an exponential number (in the number of indices) of possible designs. Indeed, it is well known that given 1) a process dependence graph (i.e., a directed acyclic graph), 2) a completion time, and 3) a number of processors, finding a multiprocessor schedule for the dag meeting the time and processor bound is NP-complete [6]. Thus, it is unlikely that an efficient algorithm exists for synthesizing space-time optimal designs, given a process dependence graph.

As a complexity measure, space-time optimality indicates how many processors are needed [sufficient] to extract the maximum amount of parallelism from an algorithm. Since this measure depends only on the [graph representation of the] algorithm, it is machine-independent.

The Aitken and the Neville algorithms, explained in §2, compute the coefficients of interpolating polynomials. Their recursions can be modified to evaluate the inter-

polating polynomial at a point  $\bar{x}$  without actually computing its coefficients. This implicit evaluation of the interpolating polynomial is named *iterated interpolation*. In this case, recursions (9) and (10) become

$$A_{ij} = \frac{(\bar{x} - x_j)A_{i-1,i} - (\bar{x} - x_i)A_{i-1,j}}{x_i - x_j} \quad (11)$$

$$N_{ij} = \frac{(\bar{x} - x_j)N_{i,j-1} - (\bar{x} - x_i)N_{i+1,j}}{x_i - x_j} \quad (12)$$

We can use these systolic arrays for iterated interpolation as follows:

1. Instead of using  $x_i$  as input, we use  $\bar{x} - x_i$ .
2. instead of programming the processors according to recursion (9) and (10), we program them according to recursions (11) and (12).

There are several extensions of the material that we have presented. We sketch some of them; a presentation of details is tedious but straightforward, hence omitted. First, the 2-level process dependence graph for computing generalized divided differences admits composite embeddings, each of which possesses its own data flow and spacetime tradeoffs. For example, the 2-level process dependence graph for computing generalized divided differences can be embedded in spacetime so that the top level graph has a spatial projection that is a 1-d array of processing elements, while the bottom level graph—the rectangular mesh process dependence graph—is embedded so that its spatial projection is a 2-d array of processing elements. Its dual embedding, a 2-d triangular array of 1-d arrays, also is possible.

Second, one can ‘compose’ our solutions for generalized divided differences and multivariate computation. That is, the schemes and arrays presented can be implemented hierarchically.

All of these design options have a place, indicating the potential usefulness of software implementations on a programmable systolic/wavefront array. Examples of such software-oriented systolic computing systems include 1) an array of Transputers<sup>3</sup> [10], 2) the Warp [1], and 3) the Matrix-1 [5].

In this paper, we have:

1. presented 3 variations on the McKeown linear systolic array, each of which is spacetime optimal;
2. presented a 2-D interpolation systolic array, and shown its applicability in the multivariate case;
3. generalized these arrays to 2-level arrays that compute generalized divided differences;
4. indicated how these techniques can be applied to Neville’s algorithm;
5. sketched the reprogramming that enables these same systolic arrays to implicitly evaluate the interpolating polynomials.

<sup>3</sup> Transputer is a trademark of INMOS, Ltd.

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