Halley's Method for the Matrix Sector Function *

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Abstract

The matrix n-sector function is a generalization of the matrix sign function, and can be used to determine the number of eigenvalues of a matrix in a specific sector of the complex plane, and to extract the eigenpairs belonging to this sector without explicitly computing the eigenvalues. It is known that Newton's method, which can be used for computing the matrix sign function, is not globally convergent for the matrix sector function. The only existing algorithm for computing the matrix sector function is based on the continued fraction expansion approximation to the principal nth root of an arbitrary complex matrix. In this paper, we introduce a new algorithm, based on Halley's generalized iteration formula for solving nonlinear equations. It is shown that the iteration has good error propagation properties and high accuracy. Finally, we give two application examples, and summarize the results of our numerical experiments comparing Newton's, the continued fraction, and Halley's method.

Technical Area: Systems, Computational methods.

Keywords: Matrix sign function, matrix sector function, eigenvalue computation, Newton's method, continued fraction method, Halley's method.

1 Introduction

Fast computation of a restricted subset of eigenpairs of time varying matrices is an important topic in real time signal processing and control applications. Approximation of a matrix by another of lower rank, or model reduction, is desired in many applications, e.g., in systems theory [25], data analysis, pattern recognition, spectral analysis, radar, sonar, and geophysics [7]. Fast sequential and parallel for computing the eigenvalues and eigenvectors of unsymmetric matrices have been developed [5, 12, 13, 15]. These algorithms can be used to compute all the eigenvalues of the matrix and then to extract the specified subset of the eigenvalue and eigenvector pairs. However, for these applications, methods which compute only a restricted subset of eigenpairs without resorting to computationally expensive eigenpair methods would be more useful and efficient. The matrix sign function and the matrix n-sector function can efficiently and reliably be used for this purpose [3, 2]. By obtaining the sector function of a matrix, we can easily determine the number of eigenvalues of a matrix in a specific sector of the complex plane, and extract the eigenpairs belonging to this sector without explicitly computing the eigenvalues and eigenvectors. By shifting the original matrix,

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or applying bilinear transformations, we can extend the sectors to various shapes and geometries. The matrix sign and sector function methods also have certain properties which make them more advantageous, e.g., complex arithmetic is avoided for matrices with real entries.

The matrix sign function is a particular case of the matrix n-sector function for $n = 2$. Sequential and parallel algorithms for the matrix sign function have been developed [21, 16, 17, 11, 20], and its applications systems theory and matrix analysis have been established [23, 24, 6, 11]. However, the matrix sector function is a fairly new research topic. To the best of our knowledge, the only existing algorithm for computing the sector function of arbitrary complex matrices is the one given in [27]. Let a matrix $A \in \mathcal{C}^{m\times m}$ have eigenspectrum $\sigma(A) = \{\lambda_i, i = 1,\ldots,m\}$ where $\lambda_i \neq 0$ and $\arg(\lambda_i) \neq \pi$. The principal nth root of A, denoted as $\sqrt[n]{A} \in \mathcal{C}^{m\times m}$, is the matrix satisfying $(\sqrt[n]{A})^n = A$ and $\arg(\sigma(\sqrt[n]{A})) \in (-\pi/n, \pi/n)$. It is known that Newton's method can be used to compute the principal nth root of a positive definite matrix [14]. However, it has been pointed out in [26, 27] that Newton's method fails to give the principal nth root of a general complex matrix, thus, cannot be used for computing the matrix sector function.

In this paper, a fast and highly accurate algorithm is proposed. We first give the definitions of the matrix sector function and point out its applications. Then we introduce Halley's method, and give a perturbation analysis which shows that the first order errors in one step do not propagate to the next step. Finally, two application examples are given, and numerical experiments comparing Newton's, the continued fraction, and Halley's methods are summarized.

2 Definition of Matrix Sector Functions

The n-sector function of a scalar (matrix) is based on the principal nth root of the scalar (matrix). We begin with the definition of the sector function for a scalar. Let $\lambda \in \mathcal{C}$ be expressed by $\lambda = \rho e^{j\theta}$, where $\rho > 0$, $j = \sqrt{-1}$, $\theta \in [0, 2\pi)$, and $\theta \neq 2\pi(k+\frac{1}{2})/n$ for $k \in [0, n-1]$. Assume that λ lies within the sector Φ_k in $\mathcal{C}$ defined as the region bounded by the sector angles $2\pi(k-\frac{1}{2})/n$ and $2\pi(k+\frac{1}{2})/n$, where $k \in [0, n-1]$. Then the scalar n-sector function of λ is defined as

$$S_n(\lambda) = e^{j2\pi k/n} \ .$$

Let $\sqrt[n]{\lambda}$ be the principal nth root of $\lambda \in \mathcal{C}$. As shown in [27], the scalar sector function of λ can be expressed as

$$S_n(\lambda) = \frac{\lambda}{\sqrt[n]{\lambda^n}} \ ,$$

where $\lambda \neq 0$ and $\arg(\lambda) \neq 2\pi(k+\frac{1}{2})/n$ for $k \in [0, n-1]$. Therefore, the scalar sector function maps a scalar in a specific sector to the bisector of the sector angles on the unit circle. The scalar sign function is a particular case of the n-sector function for $n = 2$, i.e., the complex plane is partitioned into 2 sectors: Complex numbers with positive and negative real parts are mapped to $+1$ and -1, respectively. We also define the qth n-sector function of the scalar λ for $q \in [0, n-1]$, denoted by $S_{n,q}(\lambda)$, as the transformation that takes λ to 1 if λ belongs to Φ_q and to zero otherwise:

$$S_{n,q}(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \Phi_q \ , \\ 0 & \text{otherwise.} \end{cases}$$

We can extend these definitions to complex square matrices as follows. Let $A \in \mathcal{C}^{m\times m}$ and $\sigma(A) = \{\lambda_i \ , \ i = 1,\ldots,m\}$ be its spectrum with not necessarily distinct eigenvalues $\lambda_i \neq 0$ and $\arg(\lambda_i) \neq$

Proceedings of the 12th European Conference on Circuit Theory and Design,
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$2\pi(k+\frac{1}{2})/n$ for $k \in [0, n-1]$. Let $M \in \mathcal{C}^{m\times m}$ be the modal matrix that takes A to its Jordan form as

$$A = M[J_1 \oplus J_2 \oplus \cdots \oplus J_k]M^{-1} \ ,$$

where $J_i \in \mathcal{C}^{r_i\times r_i}$ are the Jordan blocks corresponding to the ith eigenvalue with geometric multiplicity r_i, such that $\sum_{i=1}^{k} r_i = m$. Applying the matrix function definition of Giorgi [22], we can define the matrix sector function of A as

$$\begin{aligned} S_n(A) &= A\left(\sqrt[n]{A^n}\right)^{-1} \\ &= M\left[J_1\left(\sqrt[n]{J_1^n}\right)^{-1} \oplus \cdots \oplus J_k\left(\sqrt[n]{J_k^n}\right)^{-1}\right] M^{-1} \ , \end{aligned}$$

where

$$J_i\left(\sqrt[n]{J_i^n}\right)^{-1} = S_n(\lambda_i) I_{r_i} \ .$$

Therefore, the definition of the sector function of a matrix becomes

$$S_n(A) = M\left[\bigoplus_{i=1}^{m} S_n(\lambda_i)\right] M^{-1} \ .$$

Following the definition of the scalar sector function, we see that the matrix sector function maps the eigenvalues of a given matrix to the bisector of the sector angles of the corresponding region onto the unit circle while preserving the eigenvectors. Similarly, the matrix qth n-sector function of A, denoted by $S_{n,q}(A)$, is defined as

$$S_{n,q}(A) = M\left[\bigoplus_{i=1}^{m} S_{n,q}(\lambda_i)\right] M^{-1} \ ,$$

where $S_{n,q}(\lambda)$ is the scalar qth n-sector function of λ. The matrix qth n-sector function of A maps the eigenvalues of A in the sector Φ_q to 1, and the remaining eigenvalues are mapped to zero. It can be easily proven (see, Theorem 4.2 in [26]) that the matrix qth n-sector function of A is equal to

$$S_{n,q}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[S_n(A)\ e^{-j2\pi q/n}\right]^{i-1}$$

for $q \in [0, n-1]$, where $S_n(A)$ is the n-sector function of A.

3 Applications of Matrix Sector Functions

The matrix sector function can be utilized to block diagonalize a given matrix, without explicitly computing the eigenvalues and the corresponding eigenvectors. For $A \in \mathcal{C}^{m\times p}$, we define ind$[A]$ as the set of linearly independent column vectors of A. Let μ_i denote the linearly independent column vectors of $S_{n,q}(A)$ for $A \in \mathcal{C}^{m\times m}$ with nonzero eigenvalues, i.e.,

$$\mu_i = \mathrm{ind}\left[S_{n,q}(A)\right] \ \in \ \mathcal{C}^{m\times m_i}$$

for $q \in [0, n-1]$, $i \in [1, k]$, and $m = \sum_{i=1}^{k} m_i$. The block modal matrix M, defined as

$$M = [\mu_1, \mu_2, \ldots, \mu_k] \ \in \ \mathcal{C}^{m\times m} \ ,$$

Proceedings of the 12th European Conference on Circuit Theory and Design,
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can be used to block diagonalize the matrix A as

$$D = M^{-1}AM = \mathrm{diag}[A_1, A_2, \ldots, A_k] \ ,$$

where the block elements correspond to the eigenvalues in the specified sector of the complex plane. This strategy can be used to decompose a system into several smaller subsystems with similar transient characteristics. The location of the poles with respect to the sector angles determines the natural frequencies and the damping ratio of the system [1]. Decoupling with respect to the given sectors would enable us to obtain a physical realization which is more precise and stable. This analysis can be performed from both state-space and matrix-fraction description points of view. Let a q-input, p-output system be described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \ , \\ y(t) &= Cx(t) + Du(t) \ , \end{aligned}$$

where $x(t) \in \mathcal{C}^{m\times 1}$, $u(t) \in \mathcal{C}^{q\times 1}$, and $y(t) \in \mathcal{C}^{p\times 1}$. Assuming the system is observable and controllable, we can define the left and right matrix fraction description of the system as

$$\begin{aligned} H_l(s) &= C(sI - A)^{-1}B + D = D_l^{-1}(s)N_l(s) + D \ , \\ H_r(s) &= C(sI - A)^{-1}B + D = N_r^{-1}(s)D_r(s) + D \ , \end{aligned}$$

where $N_r(s)$, $D_r(s)$, $N_l(s)$, and $D_l(s)$ are polynomial matrices. Let M be the block modal matrix which block diagonalizes A, obtained using the matrix sector functions $S_{n,q}(A)$. We have

$$\begin{aligned} A_d &= M^{-1}AM = \mathrm{diag}[\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k] \ \text{for} \quad \hat{A}_i \in \mathcal{C}^{m_i\times m_i} \ , \\ B_d &= M^{-1}B = [\hat{B}_1^T, \hat{B}_2^T, \ldots, \hat{B}_k^T]^T \ \text{for} \quad \hat{B}_i \in \mathcal{C}^{m_i\times q} \ , \\ C_d &= CM = [\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_k] \ \text{for} \quad \hat{C}_i \in \mathcal{C}^{p\times m_i} \ . \end{aligned}$$

Thus, the system can be block decomposed into k subsystems as

$$\begin{aligned} \dot{x}_d(t) &= A_d x_d(t) + B_d u(t) \ , \\ y(t) &= C_d x_d(t) + Du(t) \ , \end{aligned}$$

where $x(t) = Mx_d(t)$. The input/output relationship after the decomposition is given as

$$Y(s) = (C_d(sI - A_d)^{-1}B_d + D)U(s) \ .$$

where the ith element of the transfer function matrix contains the m_i eigenvalues of the sector Φ_i.

4 Halley's Method for the Matrix Sector Function

It has been shown that Halley's generalized iteration formula for solving nonlinear equations is of third order, and its error-cubing variation converges faster than Newton's method [10, 9, 8, 4]. Halley's method can be derived by applying Newton's method to the function

$$g(s) = \frac{f(s)}{\sqrt{f'(s)}} \ ,$$

Proceedings of the 12th European Conference on Circuit Theory and Design,
Vol. 2, pages 901-904, Istanbul, Turkey, August 27-31, 1995.

which is written as

$$s_{k+1} = s_k - \frac{f(s_k)}{f'(s_k) - \frac{f''(s_k)f(s_k)}{2f'(s_k)}} \ . \tag{1}$$

We start with an alternative definition of the scalar sector function as the solution of the following equation

$$f(s) = s^n - 1 = 0 \ .$$

Solution of this equation with Halley iteration becomes

$$s_{k+1} = s_k - \frac{2s_k(s_k^n - 1)}{(n+1)s_k^n + (n-1)} \ ,$$

which reduces to

$$s_{k+1} = s_k \frac{(n-1)s_k^n + (n+1)}{(n+1)s_k^n + (n-1)} \ .$$

This iteration produces an order $[1,1]$ rational Padé approximant to

$$f(s) = \frac{s}{\sqrt[n]{1-z}} \ ,$$

where $z = 1 - s^n$. In the matrix case, $f(S_n)$ can be defined as

$$f(S_n) = S_n^n(A) - I = 0 \ .$$

Let $S_n[k]$ stand for the value of $S_n(A)$ at step k. Applying the iteration of Equation (1) to the above expression, we obtain Halley's method for matrix sector function as

$$\left.\begin{array}{rcl} S_n[0] & = & A \ , \\ S_n[k+1] & = & S_n[k] \times ((n-1)S_n^n[k] + (n+1)I) \times ((n+1)S_n^n[k] - (n-1)I)^{-1} \ , \\ \lim_{k\to\infty} S_n[k] & = & S_n(A) \ . \end{array}\right\} \tag{2}$$

Now we give a convergence analysis of Halley's method by checking the location of the eigenvalues of $S_n(A)$ as the algorithm iterates starting from $k = 0$, i.e., $S_n[0] = A$. We assume that $S_n[k]$ has an eigenvalue λ_k in the sector Φ_q at the kth step of the iteration, which can be expressed as

$$\lambda_k = \rho_k e^{j(\psi_k + 2\pi q/n)} \ ,$$

where $\rho_k = |\lambda_k|$, $q \in [0, n-1]$, and $|\psi_k| < \pi/n$. Here, λ_{k+1} can be given as

$$\lambda_{k+1} \ = \ \rho_k e^{j(\psi_k + 2\pi q/n)} \frac{(n-1)\rho_k^n e^{jn\psi_k} + (n+1)}{(n+1)\rho_k^n e^{jn\psi_k} + (n-1)} \ .$$

We expect

$$\lim_{k\to\infty} \rho_k = 1 \ \text{and} \quad \lim_{k\to\infty} \psi_k = 0 \ .$$

Let $\lim_{k\to\infty} \lambda_k$ exist and be finite. Denoting this limit by x, from Equation (2) we obtain

$$x = x \frac{(n-1)x^n + (n+1)}{(n+1)x^n + (n-1)}$$

which reduces to

$$x^{n+1} - x = 0 \ .$$

Proceedings of the 12th European Conference on Circuit Theory and Design,
Vol. 2, pages 901-904, Istanbul, Turkey, August 27-31, 1995.

Assuming $x \neq 0$, we find the solution of the limit equation as $x = \sqrt[n]{1}$, i.e., an nth root of unity.

Let z be a complex number in the sector Φ_0 which contains the first real root of unity, $+1$. For the n sector plane the following inequality should hold:

$$\left| \frac{(z-1)}{(z-z_i)} \right| < 1 \tag{3}$$

for $1 \leq i \leq n-1$, where $z_i = e^{j(2\pi i)/n}$, i.e., one of the nth roots of unity. In order to guarantee that the consecutive iterates does not pass the sector boundaries, this inequality should hold true at each step of the iteration. Let s_k be a scalar at the kth step of the iteration, then the following equality should also be satisfied for all k [19].

$$s_{k+1} - S_n(s_{k+1}) = s_{k+1} - S_n(s_k) \ .$$

The relationship between two consecutive iterates is obtained as

$$\frac{(s_{k+1}-1)}{(s_{k+1}-z)} = \frac{(s_k-1)^3}{(s_k-z)^3} \left(\frac{\sum_{j=1}^{n-1} \left(\sum_{i=1}^{j} (n-(2i-1)) s_k^{n-j-1} \right)}{\sum_{j=1}^{n-1} \left(\sum_{i=1}^{j} (n-(2i-1)) s_k^{n-j-1} z^{j-1} \right)} \right) \ ,$$

where z is one of the nth roots of unity. The first part of right hand side satisfies the convergence property (3), but the entire right hand side contains a rational term which may force the ratio to be greater than 1 for some s_k. In this case the iterate changes sectors and the iteration converges to an incorrect value. The following example was provided by Kenney and Laub [18]: Consider $z_1 = e^{2\pi/3j}$, i.e., the second cuberoot of unity. Taking $s_0 = -z_1 + 0.001 = 0.5010 - 0.8660j$ (which is in the sector Φ_0) as our initial point the first step of the Halley iteration yields $s_1 = -0.4920 + 0.8660j$, and eventually the iteration converges to z_1 rather than to 1. Such inaccuracies mostly occur for points on or near the sector boundaries.

5 Perturbation Analysis

In this section, we analyze Halley's iteration for matrix sector functions when the iterates are subject to perturbations from rounding errors at a given step k. Let $\tilde{S}_n[k] = S_n[k] + E[k]$, where $E[k]$ is the error at step k. The perturbed value of $S_n[k+1]$ can be written as

$$\begin{aligned} \tilde{S}_n[k+1] &= (S_n[k] + E[k]) \times ((n-1)(S_n^n[k] + F[k]) + (n+1)I) \times \\ &\quad ((n+1)(S_n^n[k] + F[k]) + (n-1)I)^{-1} \ , \end{aligned}$$

where

$$F[k] = E[k]S_n^{n-1}[k] + S_n[k]E[k]S_n^{n-2}[k] + \cdots + S_n^{n-1}E[k] \ .$$

Here, we have used the power expansion

$$(A+E)^n \cong A^n + EA^{n-1} + AEA^{n-2} + A^2EA^{n-3} + \cdots + A^{n-1}E$$

by ignoring the terms involving more than one error term. Assuming

$$\|(n+1)S_n^n[k] + (n-1)I\| > \|(n+1)F[k]\| \ ,$$

Proceedings of the 12th European Conference on Circuit Theory and Design,
Vol. 2, pages 901-904, Istanbul, Turkey, August 27-31, 1995.

we utilize the perturbation formula in [28] and obtain

$$(A+E)^{-1} = A^{-1} - A^{-1}EA^{-1} + O(\|E\|^2) \ .$$

Let

$$\begin{aligned} N[k] &= (n-1)S_n^n[k] + (n+1)I \ , \\ D[k] &= (n+1)S_n^n[k] + (n-1)I \ . \end{aligned}$$

Ignoring error terms of degree two or more, we obtain

$$\begin{aligned} \tilde{S}_n[k+1] &= S_n[k]N[k]D^{-1}[k] - (n+1)S_n[k]N[k]D^{-1}[k]F[k]D^{-1}[k] + \\ & (n-1)S_n[k]F[k]D^{-1}[k] + E[k]N[k]D^{-1}[k] \ . \end{aligned}$$

This gives the error expression as

$$\begin{aligned} E[k+1] &= \tilde{S}_n[k+1] - S_n[k+1] \\ &= E[k]N[k]D^{-1}[k] + (n-1)S_n[k]F[k]D^{-1}[k] - (n+1)S_n[k+1]F[k]D^{-1}[k] \ . \end{aligned}$$

Let M be the modal matrix of $S_n[k]$ and $S_n[k+1]$ such that

$$\begin{aligned} D[k] &= M^{-1}S_n[k]M &= \operatorname{diag}(\lambda_1[k], \ldots, \lambda_m[k]) \ , \\ D[k+1] &= M^{-1}S_n[k+1]M &= \operatorname{diag}(\lambda_1[k+1], \ldots, \lambda_m[k+1]) \ . \end{aligned}$$

Let $\hat{E}[k] = M^{-1}E[k]M$ and $\hat{F}[k] = M^{-1}F[k]M$, then $\hat{F}[k]$ can be written elementwise as

$$\hat{F}_{ij}[k] = \left(\sum_{l=0}^{n-1} \lambda_i^l[k] \lambda_j^{n-1-l}[k] \right) \hat{E}_{ij}[k] \ . \tag{4}$$

The matrix sector function identity suggests that

$$\lim_{k\to\infty} S_n^n[k] = I \ ,$$

thus, the elementwise error iteration becomes

$$\hat{E}_{ij}[k+1] = \hat{E}_{ij}[k] + \frac{(n-1)}{2n}\lambda_i[k]\hat{F}_{ij}[k] - \frac{(n+1)}{2n}\lambda_i[k+1]\hat{F}_{ij}[k] \ ,$$

where $\lim_{k\to\infty} \lambda_i[k] = e^{j2\pi q/n}$. Therefore, the error expression becomes

$$\hat{E}_{ij}[k+1] = \hat{E}_{ij}[k] - \frac{1}{n} e^{j2\pi q/n} \hat{F}_{ij}[k] \ . \tag{5}$$

We have two cases to consider:

1. If λ_i and λ_j are in different sectors, Equation (4) gives $\hat{F}_{ij}[k] = 0$, and the error expression (5) becomes

   $$\hat{E}_{ij}[k+1] = \hat{E}_{ij}[k] \ ,$$

   i.e., the error in the kth step is passed to, but not magnified in the $(k+1)$st step.

Proceedings of the 12th European Conference on Circuit Theory and Design,
Vol. 2, pages 901-904, Istanbul, Turkey, August 27-31, 1995.

2. If λ_i and λ_j are in the same sector, Equation (4) gives

$$\hat{F}_{ij}[k] = ne^{j2\pi q(n-1)/n} \hat{E}_{ij}[k] \ .$$

Thus, the error expression (5) becomes

$$\hat{E}_{ij}[k+1] = 0 \ ,$$

i.e., no first order error is propagated to the $(k+1)$st step.

Thus, we conclude that the first order errors in one step either have limited effect on the next step, or do not propagate to the next step at all.

6 Application Examples

Example 1

Here we analyze the state space matrix for a jet transport during cruise flight, taken from Matlab `jetdemo` (Version 4.1). The matrix A is given as

$$A = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.5980 & -0.1150 & -0.0318 & 0 \\ -3.0500 & 0.3880 & -0.4650 & 0 \\ 0 & 0.0805 & 1.0000 & 0 \end{bmatrix} .$$

In Table 1 we give the eigenvalues and their damping coefficients, natural frequencies, and locations in the complex plane.

Table 1. The eigenvalues of A and their properties.

	Newton		Continued Fraction		Halley	
μ	k	$\sigma(S_4[k])$	k	$\sigma(S_4[k])$	k	$\sigma(S_4[k])$
1	35	$\pm j$	20	1	17	1
10	47	± 1	22	$1, \pm j$	20	1
10^2	49	$1, \pm j$	26	$\pm j$	23	1
10^3	54	$\pm j$	29	$1, \pm j$	25	1
10^4	65	$\pm j$	32	$\pm j$	28	1
10^5	71	1	35	$\pm j$	31	1

Terminating the iteration when $\|S_n[k] - S_n[k-1]\| \leq 10^{-6}$ with a relative machine precision of $\epsilon = 2.2204 \times 10^{-16}$, we obtain the $S_4(A)$ as

$$S_4(A) = \begin{bmatrix} -0.0445 & -1.1338 & 0.0653 & 0.0401 \\ 0.6226 & -0.3699 & -0.0916 & -0.0306 \\ -3.8290 & -0.6083 & -0.7567 & 0.0529 \\ 1.2161 & -4.3353 & 0.3667 & -0.8289 \end{bmatrix} .$$

Proceedings of the 12th European Conference on Circuit Theory and Design,
Vol. 2, pages 901-904, Istanbul, Turkey, August 27-31, 1995.

Since there are no eigenvalues in the first sector $S_{4,0}(A)$ is computed as a zero matrix. The other three partitioned matrix 4-sector functions, i.e., $S_{4,1}(A), S_{4,2}(A), S_{4,3}(A)$ are obtained as follows:

$$S_{4,1}(A) = \begin{bmatrix} 0.4763 - 0.0014j & -0.0640 + 0.5029j & -0.0166 - 0.0492j & -0.0008 - 0.0209j \\ -0.0139 - 0.3252j & 0.3451 + 0.0301j & -0.0332 + 0.0127j & -0.0142 + 0.0011j \\ -0.6883 + 1.2262j & -1.1993 - 0.8951j & 0.1506 + 0.0289j & 0.0547 + 0.0283j \\ 1.2914 + 0.6834j & -0.8989 + 1.2688j & 0.0262 - 0.1572j & 0.0280 - 0.0576j \end{bmatrix},$$

$$S_{4,2}(A) = \begin{bmatrix} 0.0474 & 0.1281 & 0.0331 & 0.0016 \\ 0.0278 & 0.3098 & 0.0663 & 0.0284 \\ 1.3766 & 2.3986 & 0.6988 & -0.1095 \\ -2.5828 & 1.7977 & -0.0524 & 0.9441 \end{bmatrix},$$

$$S_{4,3}(A) = S^*_{4,1}(A) \ .$$

We can extract the linearly independent column vectors by using the orthogonal projection algorithm to obtain the transformation matrix M as

$$M = \begin{bmatrix} 0.4763 - 0.0014j & 0.4763 + 0.0014j & 0.0474 & 0.1281 \\ -0.0139 - 0.3252j & -0.0139 + 0.3252j & 0.0278 & 0.3098 \\ -0.6883 + 1.2262j & -0.6883 - 1.2262j & 1.3766 & 2.3986 \\ 1.2914 + 0.6834j & 1.2914 - 0.6834j & -2.5828 & 1.7977 \end{bmatrix},$$

which transforms the system matrix to three subblocks as

$$D = M^{-1}AM = \begin{bmatrix} -0.0329 + 0.9467j & 0 & 0 & 0 \\ 0 & -0.0329 - 0.9467j & 0 & 0 \\ 0 & 0 & -0.5419 & -0.9578 \\ 0 & 0 & -0.0116 & -0.0280 \end{bmatrix}.$$

The matrix D contains subblocks belonging to four sectors on the 4-sector plane. In fact, the first two diagonal elements are the complex eigenvalues of matrix A. The last block gives the real eigenvalues in Φ_2. We have calculated the relative error in the computed solution S_c with respect to the solution S obtained by explicitly computing the eigenvalues of the matrix, i.e., $\|S_c - S\|/\|S\|$, for the methods mentioned. In our experiments, Halley's method gave an error of 1.3965×10^{-15} upon termination. The continued fraction algorithm converged with a slightly larger error of 1.7294×10^{-15}, and Newton's method converged to an incorrect value. The absolute maximum off-diagonal element of matrix D is found as 5.72×10^{-5}.

Example 2

In this example we show how the impulse response of a system can be decomposed into its oscillatory and damped exponential components by utilizing the matrix sector functions. We consider a linear time invariant system, represented by the matrices

$$A = \begin{bmatrix} -2.7798 & 14.4361 & -11.9801 & -28.2392 & 27.3195 \\ -5.1596 & 28.2055 & -15.6936 & -52.2117 & 46.9724 \\ 3.4870 & -5.8394 & -9.6580 & 7.5337 & 0.1969 \\ 7.2000 & -20.0000 & -2.2000 & 27.8000 & -20.0000 \\ 12.3290 & -41.4465 & 3.2807 & 65.0112 & -50.2677 \end{bmatrix},$$

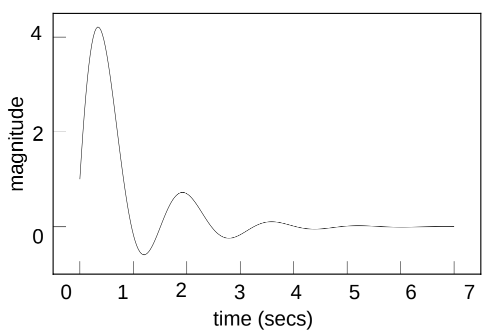
Proceedings of the 12th European Conference on Circuit Theory and Design,
Vol. 2, pages 901-904, Istanbul, Turkey, August 27-31, 1995.

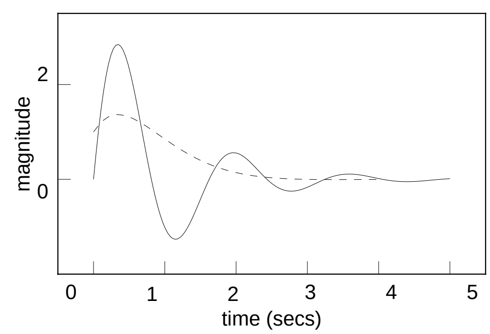
and

$$B = \begin{bmatrix} 12 & 6 & 12 & 13 & 16 \end{bmatrix}^T \quad , \quad C = \begin{bmatrix} 0.1710 & -4.7202 & 4.8860 & 9.1554 & -9.3990 \end{bmatrix} \ .$$

The eigenvalues of the open-loop system are $\sigma(A) = \{-1 \pm 3.87j, -1.5, -1.6 \pm 1.2j\}$, therefore, the impulse response has a damped exponential and an oscillatory component. Computing $S_4(A)$ enables us to decompose the system into two components $A_1 \in \mathcal{C}^{2\times 2}$ and $A_2 \in \mathcal{C}^{3\times 3}$, with eigenvalues $\sigma(A_1) = \{-1 \pm 3.87j\}$ and $\sigma(A_2) = \{-1.5, -1.6 \pm 1.2j\}$. On the 4-sector plane, the eigenvalues of A_1 lie in Φ_1 and Φ_3, where the damping ratio ξ is less than 0.707, and the eigenvalues of A_2 lies in Φ_2, where $\xi > 0.707$. Figure 1 shows the total and decomposed system impulse responses, in which the solid and dashed lines correspond to the impulse response of subsystems A_1 and A_2, respectively.

Figure 1: The total and decomposed system impulse responses.





7 Numerical Experiments

In this section, we analyze the accuracy of the three algorithms, namely, Newton's method, the continued fraction method, and Halley's method, according to the location of the eigenvalues with respect to the sector angles. We start with a diagonal matrix

$$D = \text{diag}(7 + 7j, 7 - 7j, 70 + 70j, 70 - 70j) \ ,$$

whose eigenvalues are on the sector angles $\mp\pi/4$ of the 4-sector plane. We shift the real part of this matrix by $1/\mu$ for $\mu > 0$ to obtain $D(\mu)$ as

$$D(\mu) = \text{diag}((7 + \frac{1}{\mu}) + 7j, (7 + \frac{1}{\mu}) - 7j, (70 + \frac{1}{\mu}) + 70j, (70 + \frac{1}{\mu}) - 70j) \ .$$

This changes the location of the eigenvalues of this matrix from the sector angles to Φ_0 on the 4-sector plane. After this small shift, we form the following upper triangular matrix

$$A(\mu) = D(\mu) + T$$

by adding a strictly upper triangular matrix T, with elements uniformly distributed over the interval $[0, 1]$. Now, an accurate sector function algorithm should produce $S_4(A(\mu))$ with all eigenvalues equal to 1, i.e.,

$$\sigma(S_4(A(\mu))) = \{1, 1, 1, 1\} \text{ for all } \mu > 0 \ .$$

Proceedings of the 12th European Conference on Circuit Theory and Design,
Vol. 2, pages 901-904, Istanbul, Turkey, August 27-31, 1995.

We have applied Newton's, the continued fraction, and Halley's methods to compute $S_4(A(\mu))$ for several values of μ. The results are summarized in Table 2.

Table 2. The eigenvalues of $S_4(A)$ after the convergence.

	Newton		Continued Fraction		Halley	
μ	k	$\sigma(S_4[k])$	k	$\sigma(S_4[k])$	k	$\sigma(S_4[k])$
1	35	$\pm j$	20	1	17	1
10	47	± 1	22	$1, \pm j$	20	1
10^2	49	$1, \pm j$	26	$\pm j$	23	1
10^3	54	$\pm j$	29	$1, \pm j$	25	1
10^4	65	$\pm j$	32	$\pm j$	28	1
10^5	71	1	35	$\pm j$	31	1

Our experiments shows that only Halley's iteration converges accurately for all μ. Furthermore, Halley's method computes $S_4(A)$ using fewer iterations than both Newton's and the continued fraction methods. We also note that the iterative matrix sector algorithm may use more floating-point operations than the QR algorithm, but it is easier to parallelize and contains simple matrix operations such as LU decomposition and matrix multiplication.

8 Conclusion

We have described an iterative algorithm for the computation of the matrix sector function, which is based on the solution of a nonlinear equation using Halley's method. The algorithm is fast and numerically stable, and gives accurate results even for matrices with ill-conditioned eigenstructures. We have discussed applications of matrix sector functions, and provided some examples supporting these applications. We are currently investigating the effects of scaling on the speed of convergence, and developing efficient methods for the computation of partitioned matrix sector functions from the matrix sector function.

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