

A Fast Algorithm for Matrix-Valued Nevanlinna-Pick Interpolation

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Abstract. In this paper, we derive a new fast algorithm for the matrix-valued Nevanlinna-Pick interpolation. Given n distinct points z_k in the unit disk $|z| < 1$ and n complex matrices W_k of dimension $m \times m$, satisfying the Pick condition for $1 \leq k \leq n$, the new Nevanlinna-Pick interpolation algorithm requires only $\frac{20}{3}nm^3$ complex arithmetic operations to evaluate the matrix-valued interpolatory rational function at any particular value of z , in contrast to the recently proposed algorithm which requires approximately $95nm^3$ arithmetic operations.

§1. Introduction

The classical Nevanlinna-Pick interpolation [13,14] has been shown to be a very useful tool in systems control engineering, particularly in the so-called H^∞ -control theory [3], stable systems design [1,4], circuit design [7], and system sensitivity minimization [16]. In modern mathematics, the Nevanlinna-Pick interpolation has been generalized to obtain very abstract results [3,8,9,10,15], which have also been directly and indirectly applied to systems control engineering and many other practical areas [1,3,4,7,16].

We have investigated the computational complexity and parallelism of the Nevanlinna-Pick interpolation, for the scalar-valued case in [11] and the matrix-valued case in [2]. The results which we have obtained up to now are summarized in Table 1. In this table, n is the number of interpolation points z_1, \dots, z_n and m is the dimension of the square matrix-valued function $F(z)$ that interpolates the given data W_1, \dots, W_n described in the Abstract.

The main result of the present paper is given in the last row of Table 1. As is well-known, the matrix-valued interpolation is a difficult problem in computation. However, it is also very important since it provides

optimal solutions for many multi-input/multi-output control systems that are the most demanding settings in applications. In this paper, we extend the fast computational algorithm developed in [12] from the scalar-valued Nevanlinna-Pick interpolation to the matrix-valued setting. A detailed analysis of its computational complexity is given in the last section. We are currently investigating fast parallel algorithms for this matrix-valued Nevanlinna-Pick interpolation problem.

Table 1. A Comparison of Computational Complexity

Algorithm	Operations	Processors	Reference
(Scalar NP) Sequential	$O(n^2)$	1	[11]
(Scalar NP) Parallel	$O(n)$	$O(n)$	[11]
(Scalar NP) Fast Sequential	$O(n)$	1	[12]
(Scalar NP) Fast Parallel	$O(\log n)$	$O(n)$	[12]
(Matrix NP) Sequential	$95nm^3$	1	[2]
(Matrix NP) Fast Sequential	$\frac{20}{3}nm^3$	1	This Paper

§2. Matrix-Valued Nevanlinna-Pick Interpolation Problem

We first state the matrix-valued Nevanlinna-Pick interpolation problem. Let $M(z)$ denote an arbitrary $m \times m$ matrix-valued function of the complex variable z , and define

$$H_1^\infty = \{ M(z) : M(z)M(z)^* \leq I \text{ in } |z| < 1 \},$$

where $M(z)^*$ is the complex conjugate of $M(z)$. Given n distinct points z_1, z_2, \dots, z_n in the unit disk $|z| < 1$ and n ($m \times m$) complex constant matrices W_1, W_2, \dots, W_n in H_1^∞ satisfying the Pick condition that the Pick matrix

$$P = \left[\frac{I - W_k W_\ell^*}{1 - z_k \bar{z}_\ell} \right]_{(nm) \times (nm)}$$

is nonnegative definite, find an $F(z)$ from H_1^∞ such that $F(z_k) = W_k$ for all $k = 1, 2, \dots, n$.

Note that the condition $M(z)M(z)^* \leq I$ given above is equivalent to that $M(z)$ is analytic in $|z| < 1$ and bounded in the sup-norm on $|z| = 1$.

§3. Description of the New Algorithm

Here, we extend the fast algorithm for the scalar-valued Nevanlinna-Pick interpolation developed by us in [12] to the matrix-valued case described

above, and derive a new algorithm. Define two functions: a scalar-valued function

$$a_k(z) := \frac{z - z_k}{1 - z\bar{z}_k} \tag{1}$$

and a matrix-valued function

$$B_k(Z) := [I + ZW_k^*]^{-1}[Z + W_k] \tag{2}$$

for $k = 1, 2, \dots, n$, where I is the $m \times m$ identity matrix and Z is an $m \times m$ matrix-valued complex variable (which will be substituted by some meaningful variables in the algorithm). The new algorithm for the matrix-valued Nevanlinna-Pick interpolation is the following:

The New Matrix-Valued Nevanlinna-Pick Interpolation Algorithm

Input: z, z_k , and W_k for $1 \leq k \leq n$.

Step 1: Set $F_0(z) = I$.

Step 2: For $k = 1, 2, \dots, n$, compute the $m \times m$ matrix-valued function F_k recursively by $F_k(z) = B_k(a_k(z)F_{k-1}(a_k(z)))$.

Output: $F(z) = F_n(z)$.

In the above, $a_k(z)$ is a scalar and $F_{k-1}(a_k(z))$ is an $m \times m$ matrix. Thus, the operation $a_k(z)F_{k-1}(a_k(z))$ is a scalar-to-matrix operation. The final result $F(z) := F_n(z)$ obtained in this algorithm satisfies the Nevanlinna-Pick interpolation requirement $F(z)F(z)^* \leq I$ for $|z| < 1$, with $F(z_k) = W_k$ for $k = 1, 2, \dots, n$.

Compared to the fast algorithm for the scalar-valued Nevanlinna-Pick interpolation [12], one can see that this matrix-valued interpolation algorithm has exactly the same structure, which is different by nature from the classical interpolation algorithm.

§4. Derivation of the Algorithm

Since the new algorithm is similar to the fast algorithm for the scalar-valued Nevanlinna-Pick interpolation developed in [12], the derivation can be carried out in exactly the same way as in [12], except some technical steps in handling the matrices and, in particular, matrix inversions. Recall that the main analytic tool used in [12] is the operator theoretic interpolation theorem, which holds naturally for matrix-valued functions. To verify these matrix-vector formulations and calculations, we outline the proof here for completeness of the presentation.

For the simplest case $n = 1$, it is clear that the rational function

$$F_1(z) = B_1(a_1(z)I) = \left[I + \frac{z - z_1}{1 - z\bar{z}_1} W_1^* \right]^{-1} \left[\frac{z - z_1}{1 - z\bar{z}_1} I + W_1 \right]$$

is in H_1^∞ and satisfies the interpolation condition $F_1(z_1) = W_1$. Hence, we only consider the nontrivial case where $n \geq 2$ in the following. To establish the result for the general case, however, we need some advanced mathematics.

Let V be a linear space with dual V' , and $\mathcal{L}(V)$ the space of all linear operators from V into itself. For each A in $\mathcal{L}(V)$, let A' be its adjoint, so that $(x', Ax) = (A'x', x)$ for all $x \in V$ and $x' \in V'$. As usual, a subspace $D \subseteq V'$ is called an *invariant subspace under A'* if $A'(D) \subseteq D$. The following so-called Interpolation Theorem (for the Nudel'man problem) can be found in [15].

Lemma A. Let $A \in \mathcal{L}(V)$ and $b, c \in V$ be given. Let $D \subseteq V'$ be an invariant subspace of V' under A' such that $\sum_{j=0}^\infty |(x', A^j c)|^2 < \infty$ for all $x' \in D$. Then the following two statements are equivalent:

- (i) There exists a symbol $h(z) = \sum_{j=0}^\infty z^j h_j$ in H_1^∞ such that $(x', b) = \sum_{j=0}^\infty (x', A^j c) h_j$ for all $x' \in D$.
- (ii) For all $x' \in D$, $\sum_{j=0}^\infty |(x', A^j b)|^2 \leq \sum_{j=0}^\infty |(x', A^j c)|^2$.

For the matrix-valued Nevanlinna-Pick interpolation problem given above, we may without loss of generality only consider the case where $W_k W_k^* < I$ for all $k = 1, 2, \dots, n$. The reason is that by the Pick condition we have $\frac{I - W_1 W_1^*}{1 - z_1 \bar{z}_1} \geq 0$, so that if $W_1 W_1^* = I$ then $\frac{I - W(z_1) W^*(z_1)}{1 - z_1 \bar{z}_1} = 0$. However, since the Pick matrix is Hermitian, we have

$$\left| \frac{I - W_1 W_\ell^*}{1 - \bar{z}_1 z_\ell} \right| \leq \frac{I - W_1 W_1^*}{1 - z_1 \bar{z}_1} \cdot \frac{I - W_\ell W_\ell^*}{1 - z_\ell \bar{z}_\ell}, \quad \ell = 1, 2, \dots, n.$$

Thus, $W_1 W_1^* = I$ implies that $W_\ell = W_1$ for all $\ell = 1, 2, \dots, n$. In other words, since the Pick condition is assumed, if $W_1 W_1^* = I$ then we must have $W_\ell = W_1$ for all $\ell = 1, 2, \dots, n$ in the given data, and so the constant function $F(z) = W_1$ is an expected result that satisfies both $F(z) \in H_1^\infty$ and $F(z_k) = W_k$, $k = 1, 2, \dots, n$.

To verify the algorithm for the nontrivial case where $W_k W_k^* < I$ for all $k = 1, 2, \dots, n$, we first observe that if $F(z) = F_n(z)$ is an expected result, then we must have

$$\begin{aligned} W_k &= F(z_k) = B_n(a_n(z_k) F_{n-1}(a_n(z_k))) \\ &= [I + a_n(z_k) F_{n-1}(a_n(z_k)) W_n^*]^{-1} [a_n(z_k) F_{n-1}(a_n(z_k)) + W_n], \end{aligned}$$

so that

$$a_n(z_k) F_{n-1}(a_n(z_k)) = [W_k - W_n] [I - W_n^* W_k]^{-1}$$

for all $k = 1, 2, \dots, n$. Defining

$$\tilde{z}_k = a_n(z_k) \quad \text{and} \quad \tilde{W}_k = \frac{1 - \bar{z}_n z_k}{z_k - z_n} [W_k - W_n] [I - W_n^* W_k]^{-1},$$

for $k = 1, 2, \dots, n-1$, we have $|\tilde{z}_k| < 1$, $\widetilde{W}_k \widetilde{W}_k^* < I$, and $F_{n-1}(\tilde{z}_k) = \widetilde{W}_k$, $k = 1, 2, \dots, n-1$. This means, by reversing the above procedure, that if we can find an $F_{n-1}(z)$ such that $F_{n-1}(\tilde{z}_k) = \widetilde{W}_k$, $k = 1, 2, \dots, n-1$, then we obtain

$$F(z) = B_n(a_n(z)F_{n-1}(a_n(z))),$$

which satisfies both $F(z) \in H_1^\infty$ and $F(z_k) = W_k$, $k = 1, 2, \dots, n$.

In order to prove the existence of the function $F_{n-1}(z)$, we apply Lemma A, together with the given Pick's condition. In doing so, we let $V = V' = \mathcal{C}^{(n-1)m \times m}$ be the $(n-1)$ -product of the complex space $\mathcal{C}^{m \times m}$, and we set

$$A = \begin{bmatrix} \tilde{z}_1 I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tilde{z}_{n-1} I \end{bmatrix}, \quad b = \begin{bmatrix} \widetilde{W}_1 \\ \vdots \\ \widetilde{W}_{n-1} \end{bmatrix}, \quad \text{and} \quad c = \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix},$$

where A is of dimension $(n-1)m \times (n-1)m$ while b and c are of dimension $(n-1)m \times m$. Then we have $A^j c = [\tilde{z}_1^j I \ \cdots \ \tilde{z}_{n-1}^j I]^\top$ for $j = 0, 1, \dots$. For any $a = [a_1 \ \cdots \ a_{n-1}]_{(n-1)m \times m}^\top$ and $y = [y_1 \ \cdots \ y_{n-1}]_{(n-1)m \times m}^\top$, we define $(y, a) = \sum_{k=1}^{n-1} y_k a_k$ and

$$|(y, a)|^2 = (y, a) (y, a)^* = \sum_{k=1}^{n-1} \sum_{i=1}^{n-1} y_k a_k a_i^* y_i^*.$$

Thus, for any $y \in \mathcal{C}^{(n-1)m \times m}$, we can verify that

$$\sum_{j=0}^{\infty} |(y, A^j c)|^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{n-1} \sum_{i=1}^{n-1} y_k (\tilde{z}_k^j I) (\tilde{z}_i^j I) y_i^* < \infty,$$

so that the condition stated in the lemma is satisfied.

Next, we show that the inequality in (ii) of the lemma is also satisfied. Once this is verified, we have the representation formula in (i) of the theorem, which is

$$\begin{aligned} \sum_{k=1}^{n-1} \sum_{i=1}^{n-1} y_k \widetilde{W}_k \widetilde{W}_i^* y_i^* &= |(y, b)|^2 = \sum_{j=0}^{\infty} (y, A^j c) H_j \sum_{\ell=0}^{\infty} H_\ell^* (\bar{A}^\ell c, y) \\ &= \sum_{k=1}^{n-1} \sum_{i=1}^{n-1} y_k F_{n-1}(\tilde{z}_k) F_{n-1}^*(\tilde{z}_i) y_i^* \end{aligned}$$

for all $y \in \mathcal{C}^{(n-1)m \times m}$, where $F_{n-1}(z) := \sum_{j=0}^{\infty} z^j H_j$ is a matrix-valued function of z . Clearly, this function $F_{n-1}(z)$ exists in H_1^∞ and satisfies $F_{n-1}(\tilde{z}_k) = \tilde{W}_k$ for $k = 1, 2, \dots, n-1$, since $y \in \mathcal{C}^{(n-1)m \times m}$ is arbitrary.

To show that the inequality in (ii) of the lemma holds, we note that

$$\sum_{j=0}^{\infty} |(y, A^j c)|^2 - \sum_{j=0}^{\infty} |(y, A^j b)|^2 = \sum_{k=1}^{n-1} \sum_{\ell=1}^{n-1} y_k \frac{I - \tilde{W}_k \tilde{W}_\ell^*}{1 - \tilde{z}_k \tilde{z}_\ell} y_\ell^* \geq 0,$$

by the Pick condition. Hence, we have

$$\sum_{j=0}^{\infty} |(y, A^j b)|^2 \leq \sum_{j=0}^{\infty} |(y, A^j c)|^2,$$

that is, the inequality indeed holds. This completes the derivation of the new algorithm.

§5. Computational Complexity of the New Algorithm

Given the complex numbers z_k and the $m \times m$ complex matrices W_k for $1 \leq k \leq n$, and a particular value of z , the new matrix-valued Nevanlinna-Pick interpolation algorithm computes the complex matrix $F(z)$. In order to calculate the number of arithmetic operations required to compute the matrix $F(z)$, we perform scalar-to-matrix multiplication, matrix-to-matrix addition and multiplication, and matrix inverse computations. Following the assumptions (with justifications) made in our earlier paper [2], we take the number of arithmetic operations required to perform these matrix operations as follows:

The Number of Arithmetic Operations Required by Matrix Operations

Matrix Operation	Arithmetic Operations
Scalar-to-matrix multiply	m^2
Matrix-to-matrix add	m^2
Matrix-to-matrix multiply	$2m^3 - m^2$
Matrix inverse	$\frac{8}{3}m^3 + O(m^2)$

Let $T(n)$ be the number of arithmetic operations required to compute the $m \times m$ matrix $F_n(z)$. In order to compute $F_n(z)$, the algorithm recursively calls itself. First, it computes $a_n(z)$ using the formula $a_n(z) := \frac{z - z_n}{1 - \tilde{z}_n}$, which requires 4 complex arithmetic operations. The algorithm then calls itself with $a_n(z)$ as the input to compute $F_{n-1}(a_n(z))$. According to our assumption, this requires $T(n-1)$ arithmetic operations. After the $m \times m$ matrix $X := F_{n-1}(a_n(z))$ is returned, we need to compute

$$F_n(z) = B_n(a_n(z)X) := B_n(Y) := [I + YW_k^*]^{-1}[Y + W_k].$$

This is accomplished by first computing $Y := a_n(z)X$, which requires m^2 arithmetic operations (scalar-to-matrix multiply). Then, $[I + YW_k^*]$ is computed using $2m^3 - m^2 + m$ arithmetic operations (a matrix-to-matrix multiply followed by adding 1s to the diagonal of this matrix). The inversion of the matrix $[I + YW_k^*]$ requires $\frac{8}{3}m^3 + O(m^2)$ arithmetic operations. We then perform the matrix-to-matrix addition to compute $Y + W_k$ using m^2 arithmetic operations. Finally, the matrices $[I + YW_k^*]^{-1}$ and $[Y + W_k]$ are multiplied using $2m^3 - m^2$ arithmetic operations. Summing these values, we calculate the number of arithmetic operations to compute $F_n(z)$ as

$$T(n) = T(n-1) + \frac{20}{3}m^3 + O(m^2).$$

This recursion gives the total number of arithmetic operations to compute $F(z)$ as $\frac{20}{3}nm^3$ plus the lower order terms. The algorithm proposed in this paper avoids computing matrix square-roots and inverse-square-roots, and thus much faster than the algorithm described in [2].

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References

1. Antoulas, A. C., J. A. Ball, J. Kang, and J. C. Willems, On the solution of the minimal rational interpolation problem, *Linear Algebra and Applications* **137/138** (1990), 511–573.
2. Chen, G. and Ç. K. Koç, Computing matrix-valued Nevanlinna-Pick interpolation, *Linear Algebra and its Applications* **203+204** (1994), 253–263.
3. Chui, C. K. and G. Chen, *Signal Processing and Systems Theory: Selected Topics*, Springer-Verlag, New York, 1992.
4. Chui, C. K. and G. Chen, Rational approximation theory and its applications in system engineering: An overview, *Proceedings of the 33rd Control and Decision Conference*, December 14–16, Orlando, FL (1994), 1487–1492.
5. Delsarte, Ph., Y. Genin, and Y. Kamp, Schur parametrization of positive definite block-Toeplitz systems, *SIAM J. Appl. Math.* **36** (1979), 34–46.
6. Delsarte, Ph., Y. Genin, and Y. Kamp, The Nevanlinna-Pick problem for matrix-valued functions, *SIAM J. Appl. Math.* **36** (1979), 47–61.
7. Delsarte, Ph., Y. Genin, and Y. Kamp, On the role of the Nevanlinna-Pick problem in circuit and system theory, *Circuit Theory and Applications* **9** (1981), 177–187.

8. Dym, H., *J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation*, American Mathematical Society, **71**, 1989.
9. Fedčina, I. P., A criterion for the solvability of the Nevanlinna-Pick tangent problem (in Russian), *Mat. Issled.* **7** (1972), 223–227.
10. Fenyves, F., *Beitrag zur Realisierung von Zweipolen mit Vorgegebener Charakteristik*, Atheneum, Budapest, 1938.
11. Koç, Ç. K. and G. Chen, Parallel algorithms for Nevanlinna-Pick interpolation: The scalar case, *International Journal of Computer Mathematics* **40** (1991), 99–115.
12. Koç, Ç. K. and G. Chen, A fast algorithm for scalar Nevanlinna-Pick interpolation, *Numerische Mathematik* **64** (1993), 115–126.
13. Nevanlinna, R., Über beschränkte analytische funktionen, *Annales Academiae Scientiarum Fennicae* **A32** (1929), 1–75.
14. Pick, G., Über die beschränkungen analytischer funtionen welche durch vorgegebene funktionswerte bewirkt werden, *Math. Ann.* **77** (1916), 7–23.
15. Rosenblum, M. and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford University Press, New York, 1985.
16. Vidyasagar, M., *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985.

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